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POLARIZATION OF SODIUM RESONANCE RADIATION IN MAGNETIC FIELDS. LARMOR PRECESSION OF EXCITED ATOMS.†

By A. Ellett*

Wood and the author have shown that magnetic fields of a few gauss may greatly affect the polarization of resonance radiation, the effect produced depending upon the orientation of the impressed magnetic field with respect to the electric vector of the exciting light beam. It was suggested[1] that the low initial polarization and the intense magnetic fields (200 gauss) required to produce the maximum increase or decrease in the polarization of Sodium D-line resonance radiation might be due to the presence of hydrogen. The present investigation was undertaken with the object of determining both the value of the initial polarization and the relation between polarization and intensity of the impressed magnetic field for D-line resonance radiation under more favorable experimental conditions. An improved source has been devised, with which it is possible to excite strong resonance radiation in sodium vapor at a temperature so low that the depolarizing action of high vapor pressure is entirely eliminated. With the source used in previous work it was necessary to maintain the resonance lamp at a temperature of at least 180° C in order to obtain resonance radiation of sufficient intensity for making observations of polarization. The low initial polarization (6%) was due in large part to the high vapor pressure, and to some extent to the presence of hydrogen. In the course of the present investigation it has been found that the initial polarization increases as the temperature (and vapor pressure) is

† A Preliminary report appeared in Nature, Dec. 27, 1924.

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[1] Phys. Rev. *3*, p. 243-254, 1924.

decreased until a temperature of approximately 120° C is reached. In the range between 120° C and 80° C there is no perceptible change in the value of the polarization. The observations recorded in this paper were made with the resonance lamp at temperatures between 95° and 110°C.

THE SOURCE

The source was a long discharge tube of the type used previously, Fig. 1. The central portion of the tube is run through an oven so that the side tube *S* may be maintained at a nearly constant temperature in order to secure a steady diffusion of sodium vapor into the discharge to replace that carried away by the hydrogen flow. There is a sharply marked optimum pressure of sodium for best operation of the source,

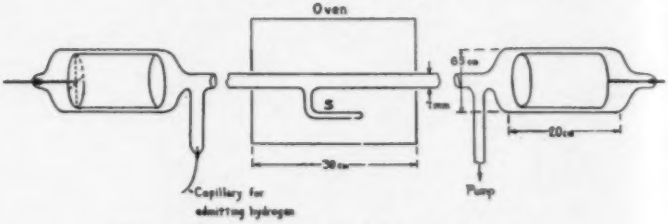

Fig. 1. *Discharge tube for exciting resonance radiation*

for a slight increase or decrease results either in loss of intensity or in reversal with loss of the center of the line, and it is only a very narrow region near the center which is effective for exciting resonance radiation in vapors at low temperature and pressure. The tube *AB* carries a current of 0.8 to 1.0 amperes under a potential drop of 3000 volts and is heated by this heavy discharge almost to the softening point of the pyrex glass of which it is made. Sodium diffusing in from the side tube is swept on down the discharge tube by the continuous current of hydrogen pumped through for that purpose. Consequently the pressure of sodium vapor in the discharge depends mainly upon the rate of diffusion of sodium out of the side tube. This diffusion is controlled by the temperature of the oven and is independent of the temperature of the discharge tube *AB*. Hence with this source the maximum input of energy is limited only by the softening point of the glass, and there is no increase of vapor pressure with high currents to cause reversal of the emission line, such as inevitably occurs with sodium and sodium amalgam arcs. The best temperature of the oven was found to be 250° C. If it was hotter than this the efficiency of the

source for excitation of resonance radiation decreased perceptibly. Observation with a 13 cm quartz Lummer plate by Hilger showed no resolvable reversal at this temperature. If the temperature was raised slightly D_2 first showed reversal at a temperature of 260°-262° and D_1 at 267°-270°. It is worthy of note that a discharge tube which had been run for well over 200 hours showed no brown discoloration such as is usually produced in sodium arcs. The same type of source has been used for exciting resonance radiation at various resonance lines of caesium, potassium, and thallium.[2]

THE RESONANCE LAMP

The resonance lamp was thoroughly evacuated and baked out before sodium, repeatedly redistilled in vacuo to free it from hydrogen, was introduced. The resonance lamp was continuously evacuated by a rapid two stage diffusion pump to which it was connected through a liquid air trap to prevent diffusion of mercury vapor into the sodium. The presence of mercury vapor reduces the intensity of the resonance light very considerably and also decreases the initial polarization.

With resonance lamps of the spherical form used previously it was very difficult to eliminate spurious light due to reflection from the curved walls of the bulb. Professor Wood suggested that rectangular flat sided bulbs might be made by suitably heating and blowing ordinary spherical bulbs. One side of the bulb is held in the flame until a circular area slightly wider than half the diameter of the bulb is softened. This side is turned uppermost and the softened glass falls in. As the heated area cools from the periphery toward the center, the bulb is slightly blown, and with a little care an almost flat side may be produced. By repetition of this process a rectangular bulb may be made. These bulbs, made from pyrex glass, may be evacuated and heated in an oven until they soften and slowly collapse without danger of their cracking. By their use errors due to scattered light are completely eliminated.

ARRANGEMENT OF APPARATUS

The arrangement of apparatus, shown diagrammatically in Fig. 2, is the same as that used by Wood and the author. Light from the source S is rendered nearly parallel by the Dunoyer Condenser D, passes through a large nicol prism and is converged at the center of the reson-

[2] Recently Paul D. Foote and the writer have performed the Cario-Franck experiment with mixtures of mercury with sodium, rubidium and caesium, using this type of source to obtain the 2536.7 mercury line with great intensity. The hydrogen streamed past a drop of mercury at room temperature before it entered the discharge.

ance lamp. Suitable shields are interposed to prevent stray light reaching the resonance lamp. The polarization was measured by compensating with inclined glass plates in the usual manner. Pringsheim has recently called attention to the fact that the polarization introduced by such plates cannot be accurately calculated from the Fresnel values of the reflection coefficients. That this is true has long been known. E. C. Pickering published curves showing the deviation of observed and computed values of polarization introduced by various numbers of plates at various angles in the Phil. Mag. for 1874, and similar observations were later discussed by the late Lord Rayleigh. In the present experiments the plates were calibrated by a modification of the Cornu method, using a soda flame as a source. The calibration was carried out for every three degrees from 5° to 45°, each point being determined as the mean of 20 concordant observations, and a smooth curve was drawn through the points so obtained.

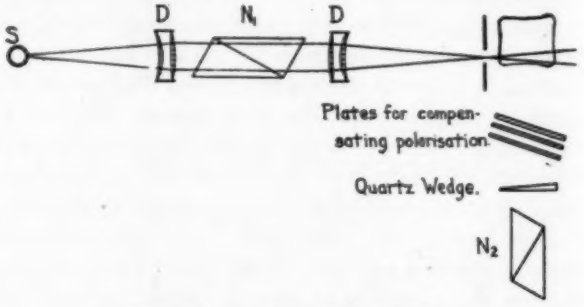

Fig. 2. *Arrangement of optical apparatus*

The accuracy of this method of determining polarization depends upon the smallest difference in intensity of illumination which the eye can detect. From the accuracy with which settings of the plates could be repeated it was estimated that a difference of one part in two hundred was readily detectable. This is based upon determinations made directly upon the resonance radiation, since with weak sources we should expect the smallest observable difference in brightness to be a larger per cent of the total intensity. The actual intrinsic brightness of the resonance radiation was not measured. It was bright enough to be seen perfectly distinctly in a well lighted room. In general it may be said that the absolute error of a single observation is about 0.5 per cent so that the mean of several observations may be

somewhat more precise. The error expressed as a percentage of the per cent polarization is of course least when the polarization is high.

POLARIZATION OF THE RESONANCE RADIATION

The per cent polarization of a partially polarized beam is defined by $\frac{P}{100}=\frac{I-I'}{I+I'}$ where I and I' are the intensities of the two plane polarized components into which the beam is resolved by a double image prism so adjusted that I/I' is a maximum. This amounts to the same thing as considering the beam resolved into a plane polarized component and an unpolarized one, P being the intensity of the plane polarized component expressed as a per cent of the total intensity. For convenience in describing the state of polarization with various orientations of the impressed magnetic field we shall consider the exciting light beam to have the direction of OX in a system of rectangular axes, Fig. 3 and shall

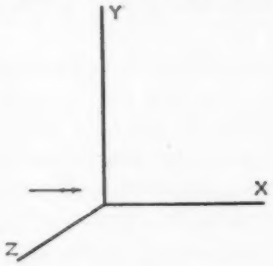

Fig. 3. *Reference axes*

record the state of polarization of the resonant light emitted along OY and OZ. First, when the exciting light beam is unpolarized, in the absence of a field, the light emitted along both OY and OZ is 9.0 per cent polarized, with its maximum electric intensity in the plane YOZ. The effect of a field parallel to OX is to increase this polarization to a maximum value of 28.0 per cent attained in a field of 60 gauss. A field along OY causes the light emitted parallel to the field to become unpolarized and produces 14.2 per cent polarization[3] in that emitted

[3] The theories of the behavior of resonance radiation in a magnetic field deduced from the quantum theory of the Zeeman effect by Breit, Joos, Hanle, and Pringsheim would lead us to expect zero polarization in this case, provided the sum of the intensities of the S components is equal to that of the P components in the Zeeman pattern. However all values of polarization observed in sodium may be computed quite closely if we assume that the intensities of the Zeeman components are in the ratios 10:4:3 in the D_2 pattern. Measurements of the intensities of the components of the D_1 Zeeman pattern do not indicate any such resultant polarization.

along OZ. The maximum intensity of the electric vector is parallel to OY. When the exciting beam is plane polarized with its electric vector parallel to OY the light emitted along OY is unpolarized in the absence of a field, while that emitted along OZ is 16.3 per cent polarized, the direction of maximum electric intensity being parallel to OY. The effect of an impressed magnetic field parallel to OY is to increase this polarization to a maximum value of 44.8 per cent reached with a field of 60 gauss. If the magnetic field is now rotated in the plane of XOY the direction of maximum electric intensity in the resonance radiation emitted along OZ rotates with it, while the percentage of polarization steadily decreases and becomes zero when the field makes an angle of $45° \pm 3°$ with OY. As the field is rotated on toward OX the light emitted along OZ again shows polarization, but the maximum electric intensity is now perpendicular to the field. With the field parallel to OX the light emitted along both OY and OZ shows 28 per cent polarization, the direction of maximum intensity of the electric vector being perpendicular to the field.

The effect of a field parallel to OZ is to rotate the plane of polarization of the light emitted along the field, the direction of rotation being that of the Larmor precession in the field. As this rotation takes place the percentage polarization decreases and ultimately becomes zero. This decrease in polarization was observed by Wood and the author, but the rotation mentioned above had not been observed previously.[4] Failure to detect the rotation may have been due to the fact that the initial polarization was very low. Fig. 4 shows the angle of rotation of the maximum electric intensity in the resonance radiation for various fields. With very small fields where the polarization has practically its initial value the azimuth of the plane of polarization can be determined quite accurately, but with increasing intensity of the field the polarization decreases to such an extent that accurate determinations of azimuth are difficult. With small fields the angle of rotation is very nearly proportional to the field. The first three points ($H < 4$ gauss) were checked repeatedly with a variation of single observations from the mean not exceeding 1°. Beyond this the error increases rapidly and may be as great as 10° for the last points observed.

It seemed that this rotation might be due to the layer of vapor between the resonating atom and the observer, but if such were the

[4] Bohr, in a note appearing in Die Naturwissenschaften, Dec. 5, 1924, states that Franck has written him of a similar observation made by Hanle with 2536.7 resonance radiation of mercury.

case the rotation should increase with depth of the layer. To test this point the exciting beam was diaphragmed down to a depth of 5 mm in the line of sight and was sent through the resonance lamp, first so

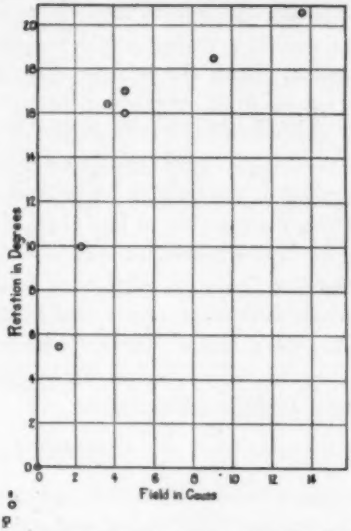

Fig. 4. *Rotation of plane of polarization as a function of the magnetic field.*

that it just cleared the front window, and then as far back from the window as the dimensions of the bulb would permit. In the first case the light from the excited atoms traversed from 1 to 6 mm of

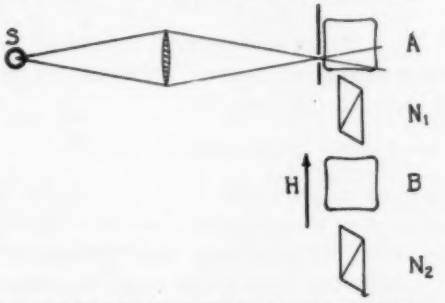

Fig. 5. *Method for demonstrating absence of Kerr effect.*

vapor, in the second case from 30 to 35 mm. The angles of rotation in various fields were the same in the two cases. An appreciable rotation in small magnetic fields of the plane of polarization of light of the resonance frequency was also looked for by another method, Fig. 5.

A is a resonance lamp excited by light from the discharge tube S. It was used to obtain light which should be as nearly as possible of the resonance frequency, and for this reason was maintained at a low temperature, 95 to 100°C. N_1 and N_2 are nicol prisms, B a bulb with flat sides, containing sodium introduced with the usual precautions for elimination of hydrogen. With the transmission planes of the nicols parallel the path of the exciting light beam through the bulb A could be seen through the prisms and the bulb B, provided its temperature was not greater than 140°C. The nicols were crossed and a magnetic field of 200 gauss applied to the bulb B. The light through the nicols was not restored, though a rotation of less than 3° should have been sufficient to restore it. The experiment was also done with the resonance lamp A in a field of the same magnitude as that applied to the bulb B, since it is conceivable that a very small departure from exact resonance might reduce the rotation. However the result was the same. These experiments show that the rotation can be explained only by an effect of the field upon the resonating atoms.

The decrease in the ratio $\frac{R}{100}\,\frac{I_y-I_x}{I_y-I_z}$ where I_y and I_z refer to intensities of components resolved parallel and perpendicular to the electric vector of the exciting light beam was also determined. The polarization is of course slightly greater than this ratio, since the direction of maximum electric intensity is not parallel to the electric vector of the exciting beam, except in the absence of a field. Values of R and H are shown Fig. 6. The equation

$$R=R_0e^{-KH}$$

with $R_0=16.3$, $K=0.10$ fits the observations quite closely. It will be remembered that a similar relation was found for the 2536.7 resonance line of mercury, with $R_0=90$, $K=1.59$.

DISCUSSION

From the observed values of the angle of rotation of the plane of polarization produced by a magnetic field we may calculate τ, the mean life of an excited atom. The average angle of rotation of an excited atom is not necessarily the same as the angle of rotation of the maximum electric intensity in the resonance radiation. If we assume that the excited atom rotates for some time t after excitation and then emits in a time very short compared to t, and assume[5] that the prob-

[5] This 'semi-classical' mechanism is able to account for the observed phenomena qualitatively, and its use as a first approximation may be justified on this account. It does not lead

ability of emission in a time between t and $t+dt$ is of the form $e^{-t/\tau}$ we find that for rotations so small that the angle of rotation and its tangent may be assumed equal we may with equal accuracy place the angle of rotation of the plane of polarization equal to the average angle of rotation of the excited atoms. From Fig. 4 we find that the limiting value of the ratio θ/H as θ tends to 0, where θ is the rotation of the plane of polarization and H, the intensity of magnetic field, is 4.7° per gauss. On the assumptions mentioned above we have $\theta=\frac{1}{2}\frac{e}{m}\frac{H}{c}\tau$. This gives $9\cdot10^{-9}$ seconds as the value of τ.

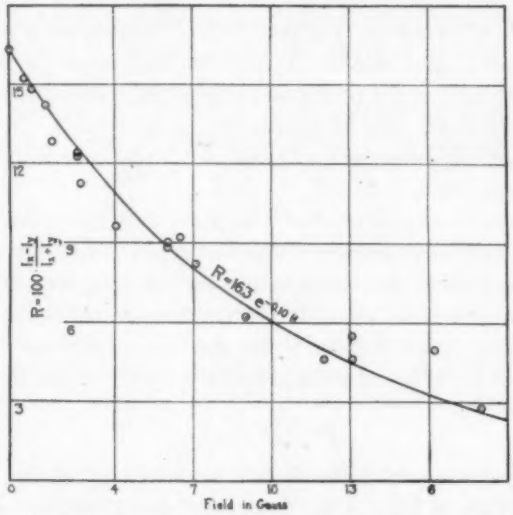

Fig. 6. *The quantity R as a function of the magnetic field*

Perhaps it may be questioned whether we should use the classical value of the Larmor precession for a line showing an anomalous Zeeman effect. If the abnormal separation of the magnetic levels of the upper quantum state is due to a nonclassical value of the Larmor precession we might better use instead of $\frac{1}{2}\frac{e}{m}\frac{H}{c}$ the expression $\frac{\Delta\nu}{\Delta\nu_n}\frac{1}{2}\frac{e}{m}\frac{H}{c}$ where

to a relation of the form $R=R_0e^{-KH}$ in accordance with observation but gives a curve having a horizontal tangent at the point $H=0$, *cf.* Eldridge, Phys. Rev. *3*, pp. 237-42; 1924, and a paper by G. Breit in the present issue of this Journal. Eldridge, in a letter to Professor Wood in April, 1924, predicted the rotation of the plane of polarization in resonance radiation. This rotation had been observed by the author previous to this time, as Eldridge notes in the paper cited above.

$\Delta\nu/\Delta\nu_n$ gives the ratio of the actual separation of the Zeeman levels in the upper state to the classical value. Where there are two or more levels in the upper state the observed rotation will have a value intermediate between that for the most and least widely separated levels. In the case of D_2 the values of $\Delta\nu/\Delta\nu_n$ for the p_1 state are 2/3 and 2. The equation for determining τ now becomes $\theta=\beta\frac{1}{2}\frac{e}{m}\frac{H}{c}\tau$, where in the case of the D-lines β is some mean value of $\Delta\nu/\Delta\nu_n$, unless the transitions which give rise to the initial polarization arise from only one of the magnetic levels of the $2p_1$ state. From the behavior of the 2536.7 resonance radiation of mercury it appears probable that the initial polarization of resonance radiation is due to transitions which in strong magnetic fields give rise to the parallel components of the Zeeman pattern. Since it is the rotation of this initial polarization with which we are concerned it may be permissible to use $\beta=2/3$. Moreover this gives $\tau=1.35\cdot10^{-8}$, whereas $\beta=2$ gives $\tau=4.5\cdot10^{-9}$, and the latter value is probably too small.

From general considerations it appears that the field required to produce a given depolarizing action will depend upon the value of the rate of precession of the excited atoms, and upon τ. Mercury resonance radiation with a large value of τ is much more readily affected by a magnetic field than is sodium D-line resonance radiation. If we write the equation for decay of polarization $P=P_0\epsilon^{-KH}$ in the form

$$P=P_0\epsilon^{-a\frac{\Delta\nu}{\Delta\nu_n}\frac{1}{2}\frac{e}{m}\frac{H}{c}\tau} \qquad (1)$$

We may evaluate a by using Wien's value of τ. For the $2p_2$ level of mercury $\Delta\nu/\Delta\nu_n$ is 3/2. The exponential[6] decay constant is 1.59, and the value[7] of τ is $1.02\cdot10^{-7}$ secs. This gives $a=1.18$.

If equation (1) is valid we should be able to calculate the polarization decay constant for D-line resonance radiation, with this value of a and the value of τ determined from the rotation of the plane of polarization of the resonance radiation. Using $a=1.18$ and $\tau=1.35\ 10^{-8}$ we find K for D-line resonance radiation to be .097, as compared with an observed value of 0.10. This agreement is suggestive, at least, and makes desirable the determination of τ and of polarization decay constants for additional resonance lines.

[6] Wood and Ellett, Phys. Rev. *3*, pp. 243-254, 1924.

[7] W. Wien, Ann. d. Phys., *73*, p. 483, 1924.

SUMMARY

1. The polarization of D-line resonance radiation with various orientations of the impressed magnetic field has been determined under experimental conditions such that the depolarizing action of high vapor pressure is eliminated. It is shown that the results are not in quantatative agreement with any of the various theories which have been advanced to account for the behavior of resonance radiation in magnetic fields.

2. It is shown that a magnetic field produces a rotation of the plane of polarization in resonance radiation, and that this rotation is due to the Larmor precession of the excited atoms. The value of τ for the $2p_1$ state of sodium is calculated from the observed rotation and found to be 1.35×10^{-8} secs.

3. It is indicated that the value of the polarization decay constant in the equation $R=R_0e^{-KH}$ may be calculated from the equation

$$K=a\frac{\Delta\nu}{\Delta\nu_n}\frac{1}{2}\frac{e}{m}\frac{1}{c}\tau.$$

where $\Delta\nu/\Delta\nu_n$ gives the ratio of the separation of the Zeeman levels in the upper state to the classical value, and a is a constant.

This investigation was carried out while in residence at the Johns Hopkins University during the academic year 1923-24.

In conclusion the author wishes to express his thanks to Professor R. W. Wood and Professor J. S. Ames for valuable suggestions and criticism.

BUREAU OF STANDARDS,
DEPARTMENT OF COMMERCE,
WASHINGTON, D. C.

Origin and Nature of the Long-range Particles from Radium C.—A zinc sulphide screen placed opposite a sample of radium C, with just enough of air or other absorbing medium placed between the two to stop all the usual alpha-particles of 7 cm range, still shows scintillations; the problem is to settle whether the particles producing them originate in the radium C, in the material behind and around it, or in the gas in front of it; and what the e/m ratio and the speed of the individual particle are. Substitution of various gases or of vacuum in the region between radium C and screen, and of various metals for the plate on which the RaC is deposited, leaves the number of these particles unaffected, and they move in straight lines from the sample of radioactive substance, which shows that they come from it. The range of the particles can be determined quite accurately by varying the pressure of gas in the container between source and screen, much less

so by interposing metal films owing to the inevitably uneven thickness of these latter. There are particles of two distinct ranges, a certain number of range 9.3 cm and about 2/11 as many of range 11 cm. The deflection of the particles of the range 9.3 cm by a magnetic field is measured and found equal to that which would be sustained by alpha-particles of equal range. These results establish that in addition to the usual mode of disintegration of RaC resulting in alpha-particles of range 7 cm, there are at least two other (much less common) modes resulting in faster alpha-particles; and they supersede the measurements of earlier observers which are supposed to have been vitiated by the solid films which they interposed between source and screen. [E. Rutherford and J. Chadwick, Cavendish; Phil. Mag. *48*, pp. 509-526; 1924.]

KARL K. DARROW

Arrangement of Atoms in the Compounds Mg_2Si and AlSb.—In each of these compounds there is a lattice of the one element interlocked with a lattice of the other. In the first instance there is a simple cubic lattice of Mg atoms interlocked with a face-centered cubic lattice of Si atoms, there being a Mg atom at the center of each of the 8 cubelets which form a single face-centered cube of the Si lattice. As Mg by itself has a close-packed hexagonal lattice and Si the diamond lattice, the combination of the two elements does violence to both; yet the distance from a Mg atom to its nearest (Si) neighbor in the lattice of the compound is accurately half the sum of the distances between adjacent Mg atoms and adjacent Si atoms in crystals of the elements, as if the Mg and the Si atoms entered as close-packed spheres of inalterable radius into each of these lattices. In the case of AlSb there seem to be two face-centered-cubic lattices interlocked, there being an atom of one at the center of each alternate cubelet of the 8 which form the face-centered cube of the other (the diamond lattice). Antimony itself crystallizes in two interlocking systems of face-centered rhombs, the center of a rhomb of either system being occupied by an atom belonging to the other; so that one may imagine that in the compound one of the two sets of Sb atoms has been slipped diagonally to the other, its atoms replaced by Al atoms, and the rhombs straightened up into cubes. (Al crystallizes in the face-centered-cubic arrangement.) But the adjacent Al and Sb atoms lie somewhat closer together than ½ the sum of the distances between adjacent Al atoms and adjacent Sb atoms in crystals of the elements. Numerical data: side of the face-centered cube of Si atoms in $Mg_2Si = 6.39$A, density of the crystal 1.95 g/cc, adjacent atoms 2.77A apart; side of the face-centered cube of either kind of atom in AlSb = 6.13A, density 4.23 g/cc, adjacent atoms 2.65A apart. Method: a sort of combination of the powder-method and the rotating crystal method, such that the reflections of different orders from a given set of atom-planes are always obtained from the same group of little crystals in the powder. [E. A. Owen and G. D. Preston, National Physical Laboratory; Proc. Phys. Soc. Lond., *36*, pp. 341-348; 1924.]

KARL K. DARROW

POLARIZATION OF RESONANCE RADIATION IN WEAK MAGNETIC FIELDS

By G. Breit

§1.

The polarization of resonance radiation has been studied by Wood,[1] Rayleigh,[2] Wood and Ellett,[3] and it has been discussed theoretically by Eldridge,[4] Pringsheim,[5] Pringsheim and Gaviola,[6] Joos,[7] and Breit.[8]

The experimental results of Wood and Ellett show that both for mercury and sodium two essentially different conditions exist, one being brought out by a sufficiently strong magnetic field and the other prevailing when the field is absent.

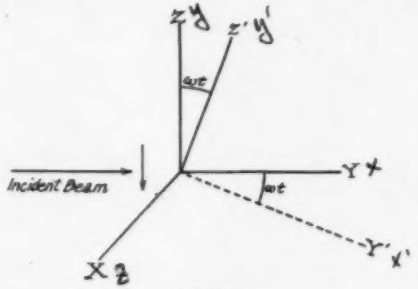

Fig. 1

Let X, Y, Z be a rectangular system of coordinates, the direction of propagation of the beam being along OX, Fig. 1. Let the magnetic field be applied along OX.

If the field is sufficiently strong, the Zeeman effect is observed along OX, the circular components being visible alone. In the absence of the field the direction of polarization of the exciting beam persists in the resonance radiation which is 90 per cent polarized for Hg and 16 per cent for D_1 and D_2 of Na. As the magnetic field is increased, the

[1] R. W. Wood, Phil. Mag. *16*, p. 184; 1908.
[2] Lord Rayleigh, Proc. Roy. Soc. (A) *102*, p. 190; 1922.
[3] Wood and Ellett, Phys. Rev. *24*, p. 243; 1924.
[4] J. A. Eldridge, Phys. Rev. *24*, p. 234; 1924.
[5] A. Pringsheim, Zeitschr. f. Phys. *23* p. 324; 1924.
[6] Gaviola and Pringsheim, ZS. f. Phys. *25*, p. 367; 1924.
[7] G. Joos, Phys. ZS. *25*, p. 130; 1924.
[8] G. Breit, Phil. Mag. *47*, p. 832; 1924.

observed polarization P depends on the applied field H in the manner expressed by the relation

$$P = P_0 e^{-KH} \tag{1}$$

where P_0 and K are constants for the particular material discussed. Thus for Hg, $P_0 = 0.90$, $K = 1.59$; for D_2 of Na, $P_0 = 0.25$, $K = 0.10$.

It is the purpose of this note to discuss the inferences which one might draw as to the mechanism of radiation from this formula.

The treatment of polarization in the absence of a magnetic field is difficult if a true model of the atom is desired in the discussion. However, it seems that in the case of mercury the atom behaves in the absence of a magnetic field very nearly as though it were an isotropic oscillator.

We shall consider first of all this idealized case. We think, therefore, of a collection of elastically bound electrons. In the presence of the magnetic field the equations of motion for this collection are the same as in the absence of the field provided they are referred to a system rotating with Larmor's precession corresponding to the magnetic field applied. Quantum theory undoubtedly is to be considered in connection with the problem. Since, however, the problem is of an essentially new type, we shall first of all give a treatment on the classical theory so as to obtain a few guiding principles in the quantum applications.

§2.

Let the angular velocity of the Larmor precession be ω. Introducing rotating axes OZ', OY', we have in these

$$E_z' = E_z \cos \omega t$$

$$E_y' = -E_z \sin \omega t \tag{1}$$

or briefly

$$E_z' + iE_y' = E_z e^{-i\omega t} \tag{1.1}$$

If in the absence of the magnetic field H the equations of motion of the oscillator are

$$\ddot{\mathbf{r}} + \kappa\dot{\mathbf{r}} + \Omega^2\mathbf{r} = \frac{\mathbf{E}}{m} \tag{2}$$

where $\mathbf{r}$ is the displacement from the position of equilibrium, then in the presence of H by Larmor's Theorem to within first order effects in H we must have

$$\ddot{\mathbf{r}}' + \kappa\dot{\mathbf{r}}' + \Omega^2\mathbf{r}' = \frac{\mathbf{E}'}{m} \tag{2.1}$$

where $\mathbf{r}'$, $\mathbf{E}'$ are referred to the rotating system of axes OX, OY', OZ'. Hence writing

$$\xi = z + iy \qquad (2.2)$$
$$\xi' = z' + iy'$$

we have

$$\ddot{\xi}' + \kappa\dot{\xi}' + \Omega^2\xi' = \frac{E_z}{m}e^{-i\omega t} \qquad (2.3)$$

Let us consider the case in which

$$E_z = E_{z0}\cos(\omega_0 t - \epsilon) \qquad (3)$$

After a steady state has been established we have for ξ

$$\xi = \xi' e^{i\omega t} = \frac{E_{z0}}{2m}\left[\left\{\Omega^2 - (\omega_0-\omega)^2 + i\kappa(\omega_0-\omega)\right\}^{-1} e^{i(\omega t-\epsilon)} + \left\{\Omega^2 - (\omega_0+\omega)^2 - i\kappa(\omega_0+\omega)\right\}^{-1} e^{-i(\omega t-\epsilon)}\right]$$

By (2.2) we have for z and y

$$z = \frac{E_{z0}}{2m}\left[\frac{\Omega^2-\omega_1^2}{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2} + \frac{\Omega^2-\omega_2^2}{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2}\right]\cos(\omega_0 t-\epsilon) + \frac{E_{z0}}{2m}\left[\frac{\kappa\omega_1}{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2} + \frac{\kappa\omega_2}{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2}\right]\sin(\omega_0 t-\epsilon) \qquad (3.1)$$

$$y = \frac{E_{z0}}{2m}\left[\frac{\Omega^2-\omega_1^2}{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2} - \frac{\Omega^2-\omega_2^2}{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2}\right]\sin(\omega_0 t-\epsilon) + \frac{E_{z0}}{2m}\left[\frac{\kappa\omega_2}{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2} - \frac{\kappa\omega_1}{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2}\right]\cos(\omega_0 t-\epsilon) \qquad (3.2)$$

where

$$\omega_1 = \omega_0 - \omega$$
$$\omega_2 = \omega_0 + \omega \qquad (3.3)$$

Thus the forced vibration is of an elliptic character. If we desire to obtain intensities of reradiated light polarized in the Z and Y directions, we must consider the quantities

$$Z(\omega_0) = \left(\frac{E_{z0}}{2m}\right)^2\left[\frac{\Omega^2-\omega_1^2}{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2} + \frac{\Omega^2-\omega_2^2}{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2}\right]^2 + \left(\frac{E_{z0}}{2m}\right)^2\left[\frac{\kappa\omega_1}{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2} + \frac{\kappa\omega_2}{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2}\right]^2 \qquad (3.4)$$

$$Y(\omega_0) = \left(\frac{E_{z0}}{2m}\right)^2\left[\frac{\Omega^2-\omega_1^2}{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2} - \frac{\Omega^2-\omega_2^2}{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2}\right]^2 + \left(\frac{E_{z0}}{2m}\right)^2\left[\frac{\kappa\omega_1}{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2} - \frac{\kappa\omega_2}{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2}\right]^2 \qquad (3.5)$$

These results express, of course, only the result of illumination with purely monochromatic light of frequency $\frac{\omega_0}{2\pi}$. Subtracting and adding (3.4), (3.5), we obtain

$$Z(\omega_0)-Y(\omega_0)=4\left(\frac{E_{x0}}{2m}\right)^2\frac{(\Omega^2-\omega_1^2)\,(\Omega^2-\omega_2^2)+\kappa^2\omega_1\omega_2}{[(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2]\,[(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2]} \tag{3.6}$$

$$Z(\omega_0)+Y(\omega_0)=2\left(\frac{E_{x0}}{2m}\right)^2\left[\left\{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2\right\}^{-1}+\left\{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2\right\}^{-1}\right] \tag{3.7}$$

If the incident radiation has a uniform spectral distribution, we can obtain the resultant values of reradiated light with polarizations in the directions of OZ, OY as

$$Z=\int_0^\infty Z\,(\omega_0)d\omega_0\,,\quad Y=\int_0^\infty Y\,(\omega_0)d\omega_0 \tag{3.8}$$

respectively. We may now note that

$$[\Omega^2-\omega_1^2+i\kappa\omega_1]^{-1}[\Omega^2-\omega_2^2-i\kappa\omega_2]^{-1}=\frac{(\Omega^2-\omega_1^2)\,(\Omega^2-\omega_2^2)+\kappa^2\omega_1\omega_2+i\kappa(\omega_2-\omega_1)\,[\Omega^2+\omega_1\omega_2]}{[(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2]\,[(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2]} \tag{3.81}$$

so that the integration of (3.6) reduces to finding the real part of the result of integrating (3.81). Using the left side of (3.81), we are dealing with an integral of the type

$$\int_0^\infty[(x-a)^2-b^2]^{-1}[(x+a)^2-b^2]^{-1}dx=\left[\frac{1}{8ab(a+b)}\log\frac{x-(a+b)}{x+(a+b)}+\frac{1}{8ab(a-b)}\log\frac{x+(a-b)}{x-(a-b)}\right]_0^\infty$$

where

$$a=\omega+\frac{i\kappa}{2}\,,\quad b=\sqrt{\Omega^2-\frac{\kappa^2}{4}}$$

In this manner it is found that

$$\int_0^\infty[\Omega^2-\omega_1^2+i\kappa\omega_1]^{-1}[\Omega^2-\omega_2^2-i\kappa\omega_2]^{-1}d\omega_0=\frac{\pi}{4}\,\frac{\frac{\kappa}{2}(\Omega^2-3\omega^2)+i\omega(\Omega^2+\frac{\kappa^2}{2}-\omega^2)}{\left(\omega^2+\frac{\kappa^2}{4}\right)[(\Omega^2-\omega^2)^2+\kappa^2\omega^2]} \tag{3.82}$$

Hence by (3.81), (3.6)

$$\int_0^\infty [Z(\omega_0)-Y(\omega_0)]d\omega_0=\pi\left(\frac{E_0}{2m}\right)^2\frac{\frac{\kappa}{2}(\Omega^2-3\omega^2)}{\left(\omega^2+\frac{\kappa^2}{4}\right)[(\Omega^2-\omega^2)^2+\kappa^2\omega^2]} \tag{3.9}$$

The integral (3.82) gives also the integrals required in the evaluation of

$$\int_0^\infty [Z(\omega_0)+Y(\omega_0)]d\omega_0.$$

For it is readily seen that

$$\int_0^\infty \left[\{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2\}^{-1}+\{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2\}^{-1}\right]d\omega_0 = 2\int_0^\infty [(\Omega^2-\omega_0^2)^2+\kappa^2\omega_0^2]^{-1}d\omega_0$$

and the latter integral is a special case of (3.82) for $\omega_1=\omega_2=\omega_0$. Hence by (3.7)

$$\int_0^\infty \left[Z(\omega_0)+Y(\omega_0)\right]d\omega_0=\frac{2\pi}{\kappa\Omega^2}\left(\frac{E_0}{2m}\right)^2 \tag{3.91}$$

Hence by (3.8)

$$P=\frac{Z-Y}{Z+Y}=\left(1+\frac{4\omega^2}{\kappa^2}\right)^{-1}\left(1-\frac{3\omega^2}{\Omega^2}\right)\left[\left(1-\frac{\omega^2}{\Omega^2}\right)^2+\frac{\kappa^2\omega^2}{\Omega^4}\right]^{-1} \tag{4.0}$$

The first factor alone is essential in this formula since $\frac{\omega}{\Omega}<<1$. Thus with great accuracy

$$P=\frac{Z-Y}{Z+Y}=\left(1+\frac{4\omega^2}{\kappa^2}\right)^{-1} \tag{4.1}$$

This is the expression obtained by Eldridge (l.c.) on his semiquantum theory. It is of interest to observe that within the approximation involved in passing from (4.0), (4.1) this form is a rigid corollary of classical dynamics as long as the incident radiation has a uniform distribution of energy among various wavelengths.

It is of interest to follow out the modifications which are introduced in (4.0) if the distribution of energy among various frequencies is non-uniform. Thus, for example, if all of the energy were concentrated into a single frequency, expressions (3.6), (3.7) would be applicable directly. With a fair approximation this would result in

$$\frac{Z(\omega_0)-Y(\omega_0)}{Z(\omega_0)+Y(\omega_0)}=\frac{(\Omega-\omega_0+\omega)(\Omega-\omega_0-\omega)+\frac{\kappa^2}{4}}{\frac{1}{2}(\Omega-\omega_0+\omega)^2+\frac{1}{2}(\Omega-\omega_0-\omega)^2+\frac{\kappa^2}{4}} \tag{4.2}$$

In a special case if $\Omega=\omega_0$

$$\frac{Z(\omega_0)-Y(\omega_0)}{Z(\omega_0)+Y(\omega_0)}=\frac{1-\frac{4\omega^2}{\kappa^2}}{1+\frac{4\omega^2}{\kappa^2}} \tag{4.21}$$

which differs from (4.1) very essentially inasmuch as it implies a diminution of $\frac{Z-Y}{Z+Y}$ to negative values and an ultimate decrease to -1.

Formulas (3.1), (3.2) enable one also to calculate the direction in the (Z, Y) plane in which the radiation appears to have a maximum of polarization. Rewriting (3.1), (3.2) as

$$\begin{aligned} z&=A_z\cos(\omega_0 t-\epsilon)+B_z\sin(\omega_0 t-\epsilon)\\ y&=A_y\cos(\omega_0 t-\epsilon)+B_y\sin(\omega_0 t-\epsilon) \end{aligned} \tag{4.3}$$

where

$$A_z=\frac{E_{z0}}{2m}\left[\frac{\Omega^2-\omega_1^2}{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2}+\frac{\Omega^2-\omega_2^2}{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2}\right] \tag{4.31}$$

$$B_z=\frac{E_{z0}}{2m}\left[\frac{\kappa\omega_1}{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2}+\frac{\kappa\omega_2}{(\Omega^2-\omega_2^2)^2-\kappa^2\omega_2^2}\right] \tag{4.32}$$

$$A_y=\frac{E_{z0}}{2m}\left[\frac{\kappa\omega_2}{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2}-\frac{\kappa\omega_1}{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2}\right] \tag{4.33}$$

$$B_y=\frac{E_{z0}}{2m}\left[\frac{\Omega^2-\omega_1^2}{(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2}-\frac{\Omega^2-\omega_2^2}{(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2}\right] \tag{4.34}$$

Taking a new system of axes $\bar{Z}$, $\bar{Y}$ obtained from Z, Y by turning it through an angle α and related to it by the equations

$$\begin{aligned} \bar{z}&=z\cos\alpha+y\sin\alpha\\ \bar{y}&=-z\sin\alpha+y\cos\alpha \end{aligned} \tag{4.4}$$

we have in that system

$$\bar{Z}-\bar{Y}=(A_z^2+B_z^2-A_y^2-B_y^2)\cos 2\alpha+2(A_yA_z+B_yB_z)\sin 2\alpha \tag{4.41}$$

$$\bar{Z}+\bar{Y}=A_y^2+A_z^2+B_y^2+B_z^2 \tag{4.42}$$

The direction of maximum polarization is the direction of maximum $\bar{Z}-\bar{Y}$ and corresponds to

$$\tan 2\alpha=2(A_yA_z+B_yB_z)\,(A_z^2+B_z^2-A_y^2-B_y^2)^{-1} \tag{4.5}$$

If the energy is distributed in the spectrum, we must integrate the numerator and denominator of (4.5). Using (4.31), (4.32), (4.33), (4.34), we have

$$A_y A_z + B_y B_z = \frac{2\kappa(\omega_2-\omega_1)\,(\Omega^2+\omega_1\omega_2)}{[(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2]\,[(\Omega^2-\omega_2^2)^2+\kappa^2\omega_2^2]}\left(\frac{E_{z0}}{2m}\right)^2 \tag{4.6}$$

$$A_z^2+B_z^2-A_y^2-B_y^2=\frac{4[(\Omega^2-\omega_1^2)\,(\Omega^2-\omega_2^2)+\kappa^2\omega_1\omega_2]}{[(\Omega^2-\omega_1^2)^2+\kappa^2\omega_1^2]\,[(\Omega^2-\omega_2)^2+\kappa^2\omega_2^2]}\left(\frac{E_{z0}}{2m}\right)^2 \tag{4.7}$$

These equations together with (4.5) give for the case of monochromatic radiation

$$\tan 2\alpha=\frac{2\kappa\omega(\Omega^2+\omega_1\omega_2)}{(\Omega^2-\omega_1^2)\,(\Omega^2-\omega_2^2)+\kappa^2\omega_1\omega_2} \tag{4.8}$$

For the case of a uniform distribution in the spectrum we integrate (4.6) by means of (3.81), (3.82)

$$\int_0^\infty (A_y A_z+B_y B_z)\,d\omega_0=\frac{\pi}{2}\,\frac{\omega(\Omega^2+\frac{\kappa^2}{2}-\omega^2)}{(\omega^2+\frac{\kappa^2}{4})\,[(\Omega^2-\omega^2)^2+\kappa^2\omega^2]}\left(\frac{E_{z0}}{2m}\right)^2 \tag{4.81}$$

and (4.7) is by (3.6) and (3.9)

$$\int_0^\infty (A_z^2+B_z^2-A_y^2-B_y^2)\,d\omega_0=\pi\,\frac{\frac{\kappa}{2}(\Omega^2-3\omega^2)}{(\omega^2+\frac{\kappa^2}{4})\,[(\Omega^2-\omega^2)^2+\kappa^2\omega^2]}\left(\frac{E_{z0}}{2m}\right)^2 \tag{4.82}$$

Hence by (4.5)

$$\tan 2\alpha=2\omega\left(\Omega^2+\frac{\kappa^2}{2}-\omega^2\right)\kappa^{-1}(\Omega^2-3\omega^2)^{-1}\cong 2\omega\kappa^{-1} \tag{4.9}$$

The curves representing (4.9), (4.1) on writing

$$\tau=\frac{\kappa}{2} \tag{5.0}$$

are represented on Figure 2.

It is seen from (4.9) that for infinite values of ω and therefore of H, α approaches 45°.

§3.

A comparison of (4.1) with the experimental results of Wood and Ellett shows that for small values of ω there is an outstanding discrepancy, the dependence of the polarization on the field being of a decidedly exponential character and decreasing very rapidly in the

case of mercury as soon as a magnetic field is applied. It appears very certain experimentally that the slope of the (P, H) curve is not zero at $H=0$. Another difference between the experimental results and the theoretical formula (4.1) lies in the fact that according to (4.1) the polarization should be complete in the absence of a magnetic field.

A possible explanation of these discrepancies may lie in the action of an internal depolarizing agent such as the existence of a precession in the atom. This might result roughly in shifting the axis of P towards higher values of H.

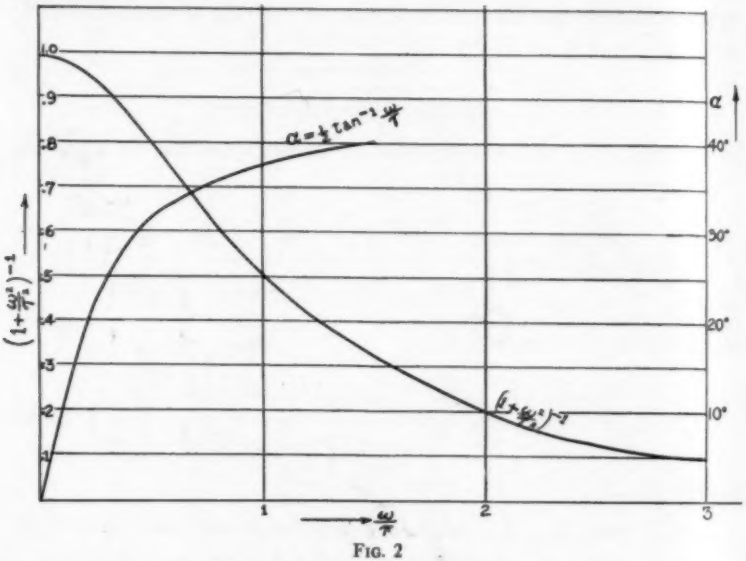

FIG. 2

However, a different cause may be seen by considering temporarily the semi-quantum model of Eldridge (l.c.). Here the absorption is imagined as taking place while a certain atomic axis is parallel to OZ. The atom precesses and turns into a new position from which it emits the light absorbed. Eldridge assumes the probability of emission in the interval $(t, t+dt)$ after absorption to be proportional to $e^{-t/\tau}$ and arrives at a result identical with our (4.1). It is not hard to find a modification of this exponential law $e^{-t/\tau}dt$ which will make the dependence of P on H exponential.

Writing

$$\omega = \beta H \qquad \left(\beta = \frac{e}{2mc}\right) \tag{5.1}$$

and denoting the probability of emission in a time lying between t and $t+dt$ after absorption by $f(t)dt$ we have on the assumption of independent phases of the emissions

$$Z=\int_0^\infty f(t)\cos^2(\beta Ht)dt \tag{5.11}$$

$$Y=\int_0^\infty f(t)\sin^2(\beta Ht)dt \tag{5.12}$$

whence

$$P=\frac{Z-Y}{Z+Y}=\int_0^\infty f(t)\cos(2\beta Ht)dt/\int_0^\infty f(t)dt \tag{5.13}$$

If P is to be equal to e^{-HK} a possible form for f is

$$f(t)=\left(t^2+\frac{\kappa^2}{4\beta^2}\right)^{-1} \tag{5.2}$$

Whereas the exponential function $e^{-t/\tau}$ has an analogy on the classical theory in the exponential law of damping, no such simple interpretation can be given to (5.2). However, it would seem that if the law of damping of the classical oscillator were modified so as to give a damping conforming to (5.2) an exponential expression for P might result.

The form (5.2) is qualitatively at least in agreement with the results of W. Wien[9] for mercury. It would appear that resonance radiation of hydrogen should show a different law of the decay of polarization from mercury because the form of $f(t)$ for it is exponential.

The main facts of the polarization in weak magnetic fields coming out of classical theory and the phenomena of polarization in strong fields being obtainable from considerations of the levels of the quantum theory, it appears that the only true treatment must be a treatment on the quantum theory which, however, takes the correspondence principle into account for weak fields in more than the customary manner. The explanation of the phenomena on the classical theory is contained in formulas (3.1), (3.2). Terms in ω_1 represent the reactions on the wave which are produced by the Zeeman component of frequency $\frac{\omega_1}{2\pi}$. These reactions are seen to interfere with the reactions due to the Zeeman component of frequency $\frac{\omega_2}{2\pi}$. Thus the interference of the reactions due

[9] W. Wien, Ann. der Physik. *73*, p. 495; 1924.

to two quantum jumps manifests itself. From this point of view it is necessary to consider actual phase relations in radiation fields, and it is also very necessary to pay attention to the spectral width of the quantum. The correspondence principle must apparently be extended to cover the case discussed above.

We are not prepared to give the exact form of this extension, but it may be suggested that such an extension will be found if use is made of the fact that the Fourier Integral representing the damped free oscillation of a classical oscillator has approximately the same energy distribution among the various frequencies as the energy distribution of forced oscillations which would be induced by these frequencies in the oscillator. (Energy is emitted during a free oscillation into the band of frequencies from which the oscillator absorbs when exposed to a homogeneous spectral distribution of radiation.)

The solution (5.2) obtained for the semiquantum theory of Eldridge suggests that the term $\kappa\dot{\mathbf{r}}$ in (2) is not quite correct, and this reminds us of the fact that damping and dissipation of energy is always a rather doubtful point physically. Fom a purely classical point of view we can always interpret damping as due to the radiation of energy by an accelerated electron. However, this radiation of energy is questioned by the quantum theory which substitutes for it sudden transitions from high to low energy states. It is natural, therefore, to seek for a completion of the theory based on equation (2) by removing the term $\kappa\dot{\mathbf{r}}$ from (2). If this is done, however, we must not forget to introduce some other means of dissipating energy. Lacking any concrete knowledge, we make the simplest hypothesis, *viz.* that the elastically bound electron is initially in a state of rest in its position of equilibrium. The incident wave disturbs the state of rest, and the electron accumulates energy according to classical laws. After a certain time of the order 10^{-7} sec., the electron is deprived of its energy by a collision or in some other way and the cycle is repeated. The time during which the electron is acted on by the wave only is supposed to be not necessarily the same during each period of accumulation of energy.

In the interval $0<t<T$ the electron is supposed to be acted on only by the wave. The equation of motion is

$$\ddot{\mathbf{r}}'+\Omega^2\dot{\mathbf{r}}'=\frac{\mathbf{E}'}{m} \tag{6}$$

Taking the incident wave to be of the form (3) we find

$$z=\frac{E_{00}}{4m}\left\{-\frac{\cos((\Omega+\omega)t-\epsilon)}{\Omega(\Omega-\omega_0+\omega)}-\frac{\cos[(\Omega+\omega)t+\epsilon]}{\Omega(\Omega+\omega+\omega_0)}-\frac{\cos[(\Omega-\omega)t+\epsilon]}{\Omega(\Omega+\omega_0-\omega)}\right.$$
$$-\frac{\cos[(\Omega-\omega)t-\epsilon]}{\Omega(\Omega-\omega-\omega_0)}+2\Big[(\Omega^2-(\omega_0-\omega)^2)^{-1}$$
$$\left.+(\Omega^2-(\omega_0+\omega)^2)^{-1}\Big]\cos(\omega_0 t-\epsilon)\right\}\quad\ldots\quad(6.1)$$

$$y=\frac{E_{00}}{4m}\left\{-\frac{\sin((\Omega+\omega)t-\epsilon)}{\Omega(\Omega-\omega_0+\omega)}-\frac{\sin((\Omega+\omega)t+\epsilon)}{\Omega(\Omega+\omega_0+\omega)}+\frac{\sin((\Omega-\omega)t+\epsilon)}{\Omega(\Omega-\omega_0-\omega)}\right.$$
$$+\frac{\sin((\Omega-\omega)t-\epsilon)}{\Omega(\Omega-\omega_0-\omega)}+2\Big[(\Omega^2-(\omega_0-\omega)^2)^{-1}$$
$$\left.-(\Omega^2-(\omega+\omega_0)^2)^{-1}\Big]\sin(\omega_0 t-\epsilon)\right\}\qquad(6.2)$$

We are concerned with the average values of $\int_0^T z^2dt$ and $\int_0^T y^2dt$, the averaging being performed over the phase ϵ. In taking the average it is useful to remember that

$$\overline{\int_0^T\cos^2(\Omega t-\epsilon)dt}=\frac{T}{2}\qquad(6.21)$$

$$\overline{\int_0^T\cos(\Omega_1 t-\epsilon_1)\cos(\Omega_2 t-\epsilon_2)dt}=$$
$$=(\Omega_1+\Omega_2)^{-1}\sin\left(\frac{\Omega_1+\Omega_2}{2}T\right)\overline{\cos\left(\frac{\Omega_1+\Omega_2}{2}T-\epsilon_1-\epsilon_2\right)}$$
$$+(\Omega_1-\Omega_2)^{-1}\sin\left(\frac{\Omega_1-\Omega_2}{2}T\right)\overline{\cos\left(\frac{\Omega_1-\Omega_2}{2}T-\epsilon_2+\epsilon_1\right)}\qquad(6.22)$$

In this expression the only term which concerns us is the last one, and this is different from zero only if $\epsilon_1=\epsilon_2$. Similarly

$$\overline{\int_0^T\sin(\Omega_1 t-\epsilon_1)\sin(\Omega_2 t-\epsilon_2)dt}=$$
$$=-(\Omega_1+\Omega_2)^{-1}\sin\left(\frac{\Omega_1+\Omega_2}{2}T\right)\overline{\cos\left(\frac{\Omega_1+\Omega_2}{2}T-\epsilon_1-\epsilon_2\right)}$$
$$+(\Omega_1-\Omega_2)^{-1}\sin\left(\frac{\Omega_1-\Omega_2}{2}T\right)\overline{\cos\left(\frac{\Omega_1-\Omega_2}{2}T-\epsilon_2+\epsilon_1\right)}\quad\cdots\quad(6.23)$$

Since we are concerned with resonance effects only, we may rewrite (6.1), (6.2) in the approximate forms

$$z=\frac{E_{x0}}{4m}\left\{-\frac{\cos((\Omega+\omega)t-\epsilon)}{\Omega(\Omega-\omega_0+\omega)}-\frac{\cos((\Omega-\omega)t-\epsilon)}{\Omega(\Omega-\omega-\omega_0)}\right.$$
$$\left.+2\left[(\Omega^2-(\omega_0-\omega)^2)^{-1}+(\Omega^2-(\omega_0+\omega)^2)^{-1}\right]\cos(\omega_0 t-\epsilon)\right\}\cdots \quad (6.3)$$

$$y=\frac{E_{x0}}{4m}\left\{-\frac{\sin((\Omega+\omega)t-\epsilon)}{\Omega(\Omega-\omega_0+\omega)}+\frac{\sin((\Omega-\omega)t-\epsilon)}{\Omega(\Omega-\omega-\omega_0)}\right.$$
$$\left.+2\left[(\Omega^2-(\omega_0-\omega)^2)^{-1}-(\Omega^2-(\omega^0+\omega)^2)^{-1}\right]\sin(\omega_0 t-\epsilon)\right\}\cdots \quad (6.4)$$

Using (6.21), (6.22), (6.3), (6.4), we have on letting

$$\Omega+\omega=\Omega_1\,,\quad \Omega-\omega=\Omega_2 \quad (6.5)$$

$$\left(\frac{4m}{E_{x0}}\right)^2\overline{\int_0^T(z^2-y^2)dt}=\frac{8T}{(\Omega_1^2-\omega_0^2)(\Omega_2^2-\omega_0^2)}+\frac{2\sin(2\omega T}{(2\omega)\Omega^2(\Omega_1-\omega_0)(\Omega_2-\omega_0)}$$
$$-\frac{4}{\Omega(\Omega_1-\omega_0)(\Omega_2-\omega_0)}((\Omega_1^2-\omega_0^2)^{-1}\sin(\Omega_1-\omega_0)T-(\Omega_2^2-\omega_0^2)^{-1}\sin(\Omega_2-\omega_0)T) \quad (6.6)$$

$$\left(\frac{4m}{E_{x0}}\right)^2\int_0^T(z^2+y^2)dt=T\left[\Omega^{-2}((\Omega_1-\omega_0)^{-2}+(\Omega_2-\omega_0)^{-2})+4(\Omega_1-\omega_0)^{-2}\right.$$
$$\left.(\Omega_2+\omega_0)^{-2}+4(\Omega_1+\omega_0)^{-2}(\Omega_2-\omega_0)^{-2})\right]-4\Omega^{-1}\left[(\Omega_2+\omega_0)^{-1}\right.$$
$$\left.(\Omega_1-\omega_0)^{-1}\sin(\Omega_1-\omega_0)T+(\Omega_1+\omega_0)^{-1}(\Omega_2-\omega_0)^{-1}\sin(\Omega_2-\omega_0)T)\right] \quad (6.7)$$

Dropping terms of higher order in $\Omega-\omega_0$ this is

$$\left(\frac{4m}{E_{x0}}\right)^2\overline{\int_0^T(z^2-y^2)dt}=2\Omega^{-2}(\Omega_1-\omega_0)^{-1}(\Omega_2-\omega_0)^{-1}$$
$$\left[T+\frac{\sin 2\omega T}{2\omega}-\frac{\sin(\Omega_1-\omega_0)T}{\Omega_1-\omega_0}-\frac{\sin(\Omega_2-\omega_0)T}{\Omega_2-\omega_0}\right]\cdots \quad (6.8)$$

$$\left(\frac{4m}{E_{x0}}\right)^2\overline{\int_0^T(z^2+y^2)dt}=2\Omega^{-2}(\Omega_1-\omega_0)^{-2}\left[T-\frac{\sin(\Omega_1-\omega_0)T}{\Omega_1-\omega_0}\right]$$
$$+2\Omega^{-2}(\Omega_2-\omega_0)^{-2}\left[T-\frac{\sin(\Omega_2-\omega_0)T}{\Omega_2-\omega_0}\right] \quad (6.9)$$

Multiplying these expressions by $d\omega_0$ and integrating from 0 to ∞ in order to take into account the effect of a uniform spectral distribution, we obtain neglecting terms of higher order in Ω^{-1}

$$\left(\frac{4m}{E_{x0}}\right)^2\int_0^\infty\left(\int_0^T(z^2-y^2)dt\right)d\omega_0=2\pi\frac{\sin^2(\omega T)}{\Omega^2\omega^2} \quad (7.0)$$

$$\left(\frac{4m}{E_\infty}\right)^2 \int_0^\infty \left(\int_0^T (z^2-y^2)dt\right) d\omega_0 = \frac{2\pi T^2}{\Omega^2} \tag{7.1}$$

If all values of T should be equal, we should have

$$\frac{Z-Y}{Z+Y} = \frac{\sin^2(\omega T)}{\omega^2 T^2} \tag{8}$$

If the probability of T lying between T and $T+dT$ should be proportional to

$$e^{-T/T_0}$$

then we find by means of (7.0), (7.1) that

$$\frac{Z-Y}{Z+Y} = \frac{1}{1+4\omega^2 T_0^2} \tag{9}$$

This is, of course, in agreement with (4.1) and with the result of Eldridge (l.c.) to a certain extent.

In general, however, the point of view brought out by formula (6) may lead to very different results. Thus if the number of atoms having times between T and $T+dT$ should be

$$dN = f(T)dT \tag{10}$$

then

$$\frac{Z-Y}{Z+Y} = \int_0^\infty \omega^{-2}\sin^2(\omega T) f(T)dT \Big/ \int_0^\infty T^2 f(T)dT \tag{11}$$

If this should be equal $e^{-\omega\omega}$ as Ellett's results would suggest, the solution $\frac{3T^2-\left(\frac{a}{2}\right)}{\left(T^2+(\frac{a}{2})^2\right)^3}$ for $f(T)$ suggests itself. However, to this solution there is the objection that it is negative for small values of T. Thus a theory postulating this form of $f(T)$ cannot have a real significance. However, there may be other forms of $f(T)$, and further the derivation leading to formula (11) can be considered only as tentative because the phase relations between various atoms are not considered.[10] Even so, it is clear that (11) is more general than (9) and that, therefore, the point of view of interrupted oscillations, while yielding in a special case the

[10] The writer has developed a treatment of phenomena of dispersion in the absorption band which shows that such relations must be considered and that if they are taken into account a satisfactory explanation of the finite refractive index is obtained without making

same results which are given by the point of view of forced vibrations, gives in addition possibilities of accounting for the disagreement of experimental results with the purely classical point of view.

DEPARTMENT TERRESTRIAL MAGNETISM,
CARNEGIE INSTITUTION OF WASHINGTON,
WASHINGTON, D. C., Dec. 23, 1924.

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use of a frictional term in the equation of motion of the resonator. (See J. Wash. Acad., 15 p. 3G, 1925.)

In fact if such a calculation is performed then it is found that an interruption of length T contributes to Z and Y on the average the amounts

$$z = \frac{E_{\omega}}{4m\Omega}\left\{\left(\frac{T}{\Omega_1-\omega_0}-\frac{\sin(\Omega_1-\omega_0)T}{(\Omega_1-\omega_0)^2}+\frac{T}{\Omega_2-\omega_0}-\frac{\sin(\Omega_2-\omega_0)T}{(\Omega_2-\omega_0)^2}\right)\cos(\omega_0 t-\epsilon)\right.$$
$$\left.+2\left(\frac{\sin^2\left(\frac{\Omega_1-\omega_0}{2}T\right)}{(\Omega_1-\omega_0)^2}+\frac{\sin^2\left(\frac{\Omega_2-\omega_0}{2}T\right)}{(\Omega_2-\omega_0)^2}\right)\sin(\omega_0 t-\epsilon)\right\}$$

$$y = \frac{E_{\omega}}{4m\Omega}\left\{\left[\left(\frac{T}{\Omega_1-\omega_0}-\frac{\sin(\Omega_1-\omega_0)T}{(\Omega_1-\omega_0)^2}\right)-\left(\frac{T}{\Omega_2-\omega_0}-\frac{\sin(\Omega_2-\omega_0)T}{(\Omega_2-\omega_0)^2}\right)\right]\sin(\omega_0 t-\epsilon)\right.$$
$$\left.-2\left(\frac{\sin^2\left(\frac{\Omega_1-\omega_0}{2}T\right)}{(\Omega_1-\omega_0)^2}-\frac{\sin^2\left(\frac{\Omega_1-\omega_0}{2}T\right)}{(\Omega_2-\omega_0)^2}\right)\cos(\omega_0 t-\epsilon)\right\}$$

When these values are averaged by multiplying by $e^{-T/T_0}d\left(\frac{T}{T_0}\right)$ we obtain

$$\bar{z} = \frac{E_{\omega}}{4m\Omega}T_0^2\left\{\left(\frac{T_0(\Omega_1-\omega_0)}{1+T_0^2(\Omega_1-\omega_0)^2}+\frac{T_0(\Omega_2-\omega_0)}{1+T_0^2(\Omega_2-\omega_0)^2}\right)\cos(\omega_0 t-\epsilon)\right.$$
$$\left.+\left(\frac{1}{1+T_0^2(\Omega_1-\omega_0)^2}+\frac{1}{1+T_0^2(\Omega_2-\omega_0)^2}\right)\sin(\omega_0 t-\epsilon)\right\}$$

$$\bar{y} = \frac{E_{\omega}}{4m\Omega}T_0^2\left\{\left(\frac{T_0(\Omega_1-\omega_0)}{1+T_0^2(\Omega_1-\omega_0)^2}-\frac{T_0(\Omega_2-\omega_0)}{1+T_0^2(\Omega_2-\omega_0)^2}\right)\sin(\omega_0 t-\epsilon)\right.$$
$$\left.-\left(\frac{1}{1+T_0^2(\Omega_1-\omega_0)^2}-\frac{1}{1+T_0^2(\Omega_2-\omega_0)^2}\right)\cos(\omega_0 t-\epsilon)\right\}$$

This corresponds to (3.1), (3.2) of the damped vibrator treatment. Hence the result must be also the same so far as P and a are concerned.

Dr. L. Silberstein has predicted in a general way many of the phenomena discussed here, a number of years ago (Phil. Mag. *32*, p. 273 1916).

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Luminous Discharges in Gases at Low Pressure Placed Near an Induction Coil.—These discharges, which are observed in tubes whether with or without internal electrodes when brought close to an induction coil producing 5 cm or longer sparks, are of two types: a diffused luminosity, and a crooked train of luminous spherules. Various simple experiments indicate that the diffused luminosity occurs when the gas is exposed to a rapidly alternating electromagnetic field and is quenched when brought into a region where the potential remains constant, such as the vicinity of the earthed end of the secondary coil. The discharge is most intense at a pressure of about 10^{-2} mm and is affected by bringing a magnet close to the tube. [D. B. Deodhar, Lucknow; ZS. f. Phys. *25*, pp. 338-341; 1924.]

KARL K. DARROW

INSTRUMENT SECTION

SOME NEW RECORDING INSTRUMENTS*

By Robert S. Whipple

The development of all engineering instruments is towards automatic recording, and this is especially desirable when phenomena which are changing continually have to be measured. In many cases it is essential that the instantaneous value of a quantity should be measured rather than the mean or the maximum value. As the apparatus may have to be used in positions where it is exposed to the weather, and where it is to be handled by comparatively unskilled observers, it is important that the recording mechanism shall be as simple and robust as possible.

In the first three instruments described in this paper, the essential feature is the method of recording due to Mr. W. G. Collins. Records of very small size are obtained by the action of a moving stylus upon transparent celluloid film. The pressure upon the stylus is extremely small, and the celluloid appears to flow plastically about the rounded point of the stylus. The records thus obtained on the film can be read by a microscope accurately to 0.01 mm or can be projected optically, so that considerable optical magnification may be obtained (frequently up to 50 diameters), which, with an initial mechanical amplification of ten times, gives a total magnification of 500 diameters. As small movements can thus be successfully recorded the moving parts of the instruments used can be made light with low velocities, so that, even at high frequencies, periodic phenomena are recorded accurately and are undisturbed by inertia effects. The record on celluloid has also the advantage of durability in that it cannot fade and is non-hygroscopic. This method of recording was, I believe, first used on a differential torque indicator and, later, on an engine indicator for indicating high-speed internal combustion engines. The micro-indicator (as it is called) has now been on the market for about two years.[1]

Stress Recorder

This instrument, as its name implies, has been designed to measure the stresses set up in members of a structure when a load is placed upon

* A paper read at Toronto before the Engineering Section of the British Association, 1924.

[1] See proceedings of Inst. of Mech. Engs. 2, p. 127, 1923.

the structure. Fig. 1 is a diagrammatic section of the instrument. The recorder stands on three toes (A, B and a toe immediately behind A) made of a suitably heat-treated alloy steel. A cramp is placed upon the projecting part C of a spring plunger, which yields to the cramp so that the instrument is held on to the member or girder, the extension of which it is wished to measure, by a pre-determined pressure. The instrument is, in fact, an extensometer, the extension or compression of the member between the points A and B being measured when the load is placed upon it. The point B is carried on the part D, which is free to move in a direction parallel to the length of the instrument. This movement can take place because the pillars E and E_1 are reduced at the points marked, the reduction in the size of the pillars allowing them to bend at these points, thus forming hinges. The stylus

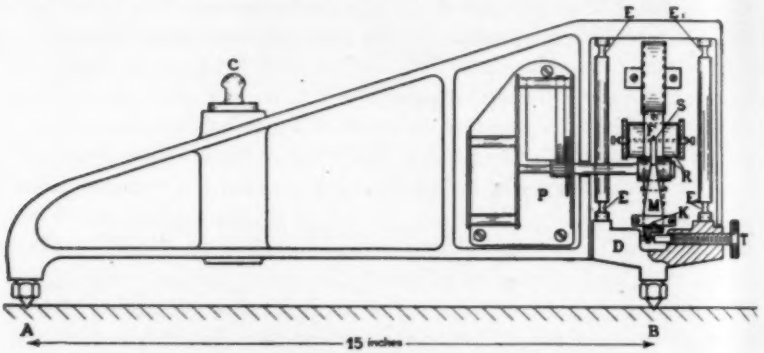


FIG. 1. *Diagrammatic section of Stress Recorder.*

S is carried by the arm M, which is supported on the knife-edge K, a similar knife-edge, not shown in the diagram, helping to support it at the back. The arm is kept in contact with the knife-edges by a spring not shown in the diagram. The zero position of the stylus can be adjusted by means of the screw T. The celluloid film F is made to traverse under the stylus by means of the rollers R, and the clock-work movement P; the speed of movement of the film can be varied (by an adjustable governor of novel design) from about 3 to 20 mm per second. Two electromagnetic marking mechanisms, not shown in the diagram, are fitted, so that a time scale, generally tenths of a second, can be marked on the film by one mechanism and the position of the moving load, or some other synchronizing mark, recorded by the other. In each case this is done by means of an additional stylus operating on the back of the

film so that it is possible for the stylus making the record to pass over the whole of its range without fouling either the time-marking or the

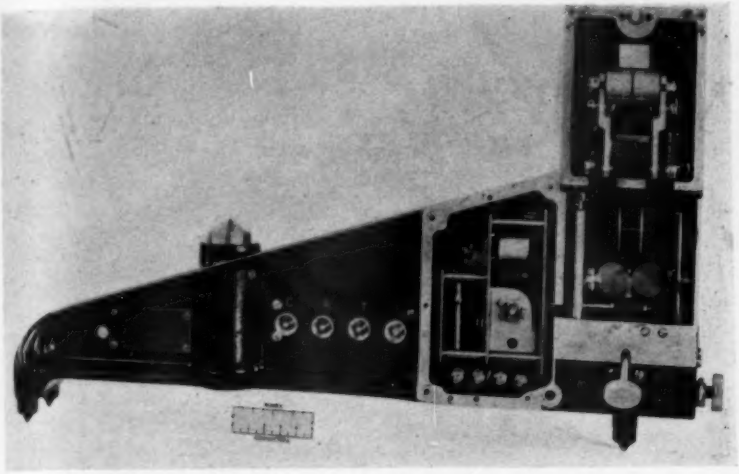


FIG. 2. *Stress Recorder, showing internal mechanism.*

position-recording stylus. The functioning of the instrument may be easily understood. Assume that the girder on which the instrument is

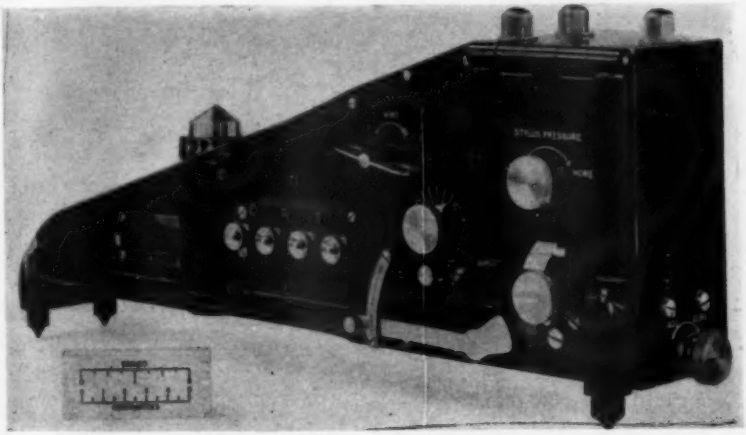


FIG. 3. *Stress Recorder, Case Closed.*

fixed, is compressed. Then the distance between *A* and *B* is shortened by a small amount, and the part *D* approaches nearer to the main body

of the instrument owing to the yielding of the pillars at E and E_1. The relative movement of these two parts causes the stylus to move over the travelling film so that a continuous record is obtained. The instrument records the changes of stress in a girder, i.e., the increase in stress due to a live load, and will not measure the dead load or the total stress due to live and dead loads. The live load stress is necessarily small so that the actual extensions to be measured, with say an accuracy of 0.1 ton per square inch, with an instrument having a base length of 15 inches, are only two or three thousandths of an inch and these must be measured to 0.0001 inch. The mechanical magnification of the record in the instru-

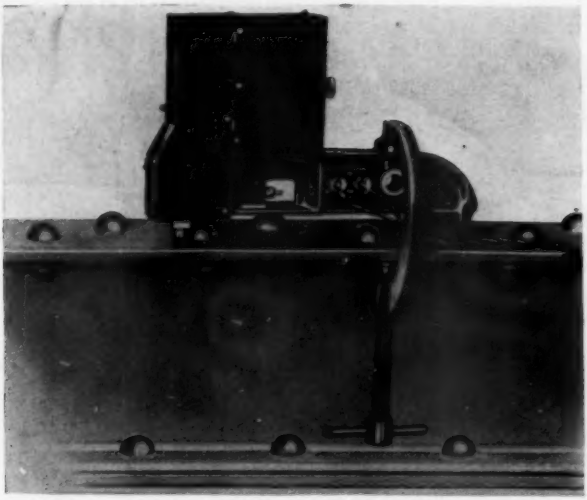


FIG. 4. *Stress Recorder clamped to a girder.*

ment is ten, and in taking photographic enlargements this is usually increased optically from 15 to 30 diameters, thus giving a total linear magnification of the record from 150 to 300. If the behavior of a structure is to be examined under live loads, it is generally advisable to have three or four instruments placed on various members of the structure. The second electric marker fitted to the instrument enables a mark to be made simultaneously on all the records, thus enabling the records to be compared with each other without difficulty. Not infrequently the locomotive operates this marker. . Electric contacts are sometimes fitted so that the clock-work mechanism can be started simultaneously on all the instruments, either automatically by the locomotive, or by

FIG. 5. *Stress Recorder clamped to a girder and electrically operated from a distance.*

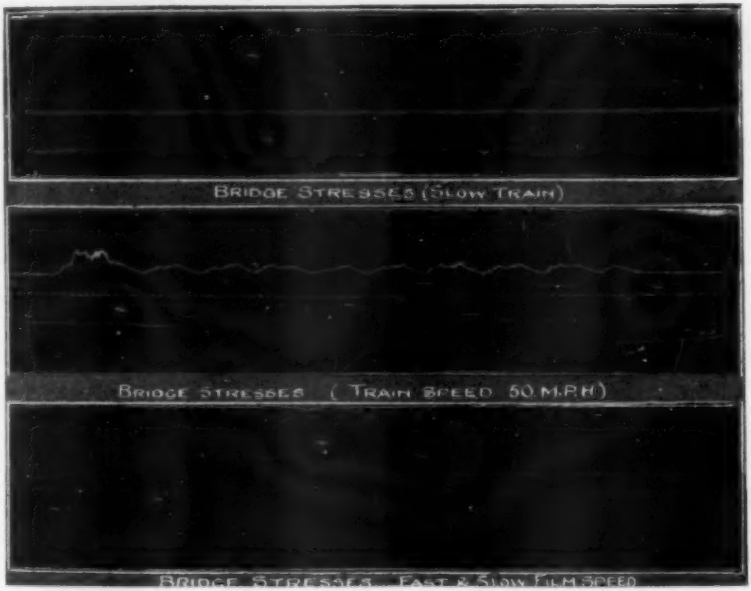


FIG. 6. *Records obtained with a Stress Recorder showing stresses set up when a train crossed a railway bridge.*

an observer who is in charge of the experiments. A control of this kind prevents waste of film, and enables the observer to leave the instruments in fairly inaccessible positions during an experiment. Figs. 2 and 3 show the complete instrument. Fig. 4 is an illustration of an instrument clamped on to a girder under test. Fig. 5 shows an observer electrically controlling the stress recorder. In Fig. 6 are reproduced some records of stresses set up in a member of a railway bridge due to the passing of trains at different speeds. In the lowest record the film speed was increased approximately five-fold (by a simple mechanism) during part of the test.

The instrument can be calibrated by applying known loads to a girder on which it is mounted. As will be seen from the illustrations it is portable and can be used in any position. The record reproduced in Fig. 6a is of interest inasmuch as it shows the sensitivity of the instru-

FIG. 6a

ment, and also from the small damping it is seen that there is little friction between the stylus and the film. The instrument was clamped near the middle of a large steel bar which was supported at its mid-point and was lightly struck with an 8 oz. hammer at one end. The bar vibrated longitudinally at a frequency of approximately 1300 per second, and the record clearly shows the oscillations: it is practically a record of the velocity of sound in steel.

VIBROGRAPH

This instrument is a form of seismograph in which the records are inscribed upon celluloid. Fig. 7 is a diagrammatic section. In the instrument for recording vertical vibrations the mass M (about 16 lbs.), is supported by means of a metal strip P from a knife-edge K, which also carries the stylus arm, and a spring Q acting through a bell-crank lever L. When not in use the mass M is fixed in position by means of the clamp E and the fixed part F, but when in use the clamp E is released and the weight of the mass is taken by the strip P. In this position the weight is free and by its inertia remains at rest. If the

vibrograph is then placed on a support which is vibrating vertically, the body of the instrument moves relatively to the weight, the celluloid film which is wrapped around the drum *D* passes beneath the stylus, and a record of the vibration is made on the celluloid. The drum is driven by means of clock-work mounted in the case *C*; the speed of the movement of the film can be varied from 3 to 14 mm per second.

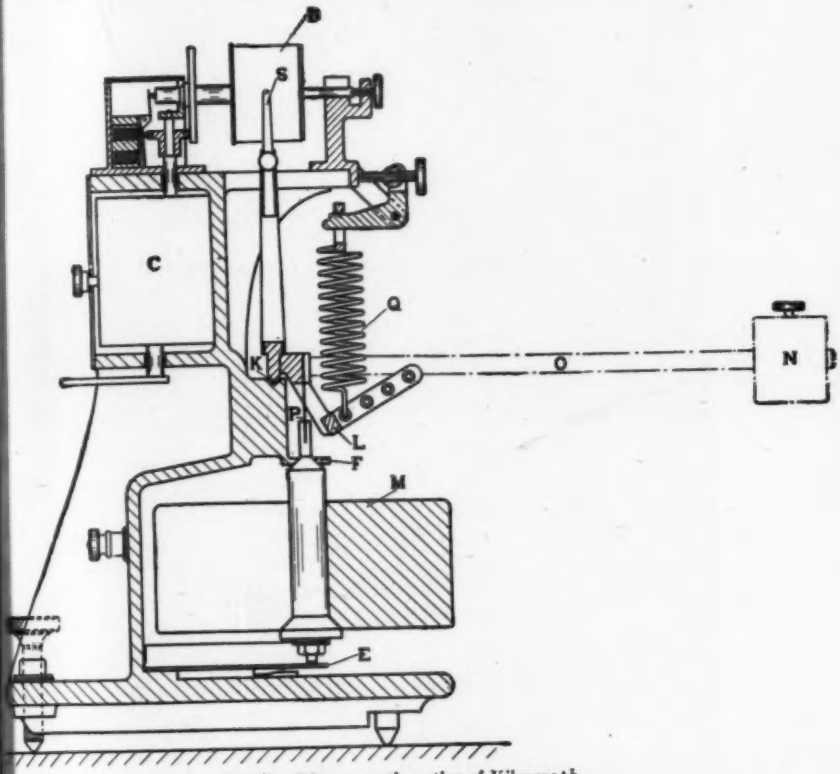


FIG. 7. *Diagrammatic section of Vibrograph.*

Electrical mechanism is provided for time marking, or for synchronizing the instrument with other instruments. The magnification by the lever is ten times the actual relative movement of the mass and the frame. By adjusting the spring *Q* the periodicity of the moving system can be varied from about 0.5 to 1 vibration per second. The optical magnification is generally about 30, so that the total magnification of a record is approximately 300. Where vibrations of considerable amplitude are

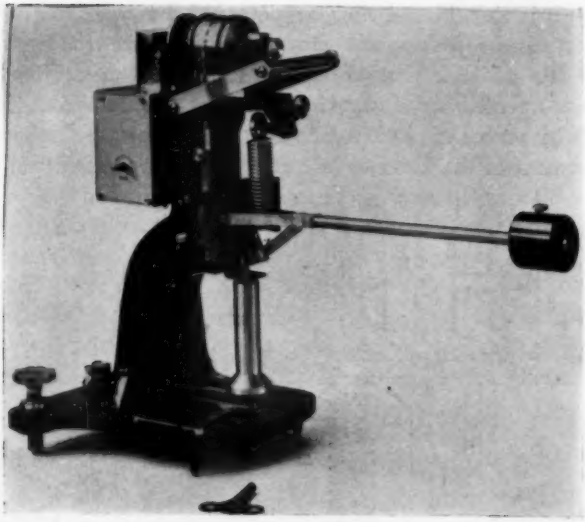


FIG. 8. *Vibrograph—arranged for recording large vibrations*

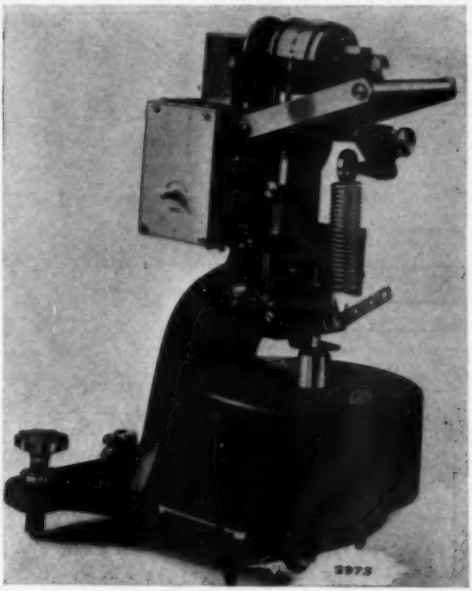


FIG. 9. *Vibrograph—arranged for recording small vibrations*

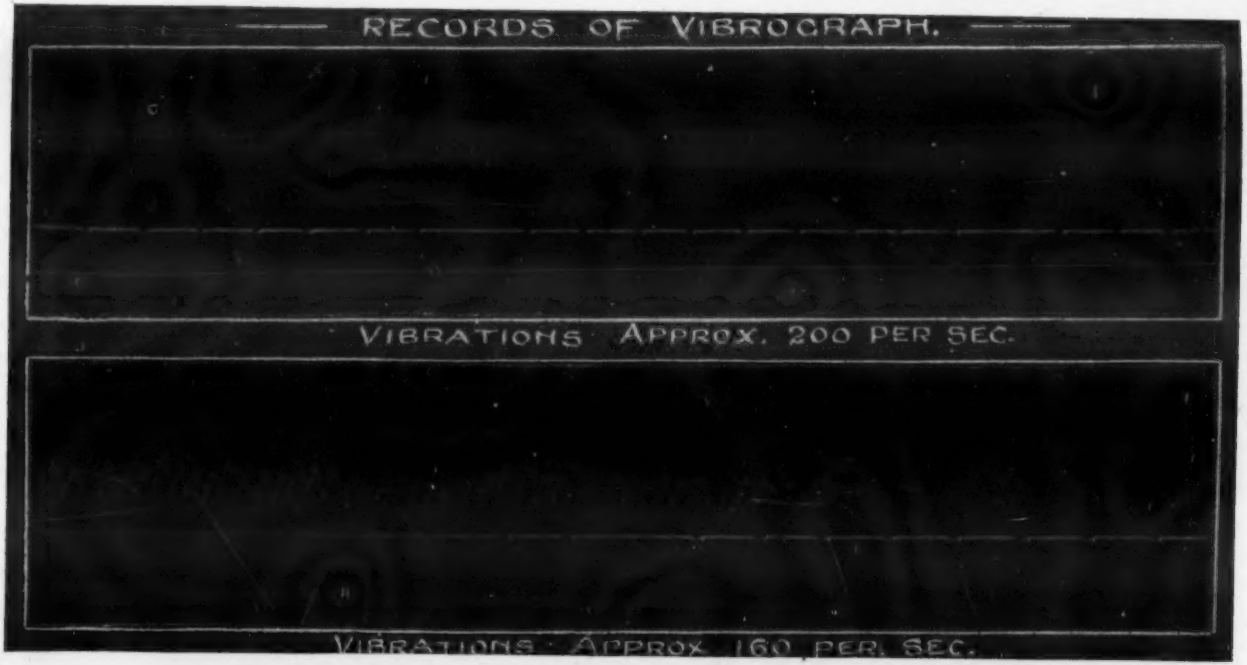


FIG. 10. *Records obtained with a Vibrograph when the instrument was placed near a steam turbine.*

to be measured, a smaller weight N, the position of which can be varied on a lever arm O, is used. This enables unmagnified records, or, in some cases, diminished records to be taken. Figs. 8 and 9 show the instrument with and without the smaller weight. Fig. 10 shows reproductions of records obtained when the instrument was placed near a steam turbine. The amplitudes of the vibrations were approximately

FIG. 11. *Record obtained with a Vibrograph showing road vibrations produced by a motor bus*

0.06 and 0.03 mm. respectively. The record reproduced in Fig. 11, shows the vibrations set up in a road by a motor bus. The main shocks were made when the front and back wheels of the bus were passing over a 3×0.5 inch board, four feet away from the instrument.

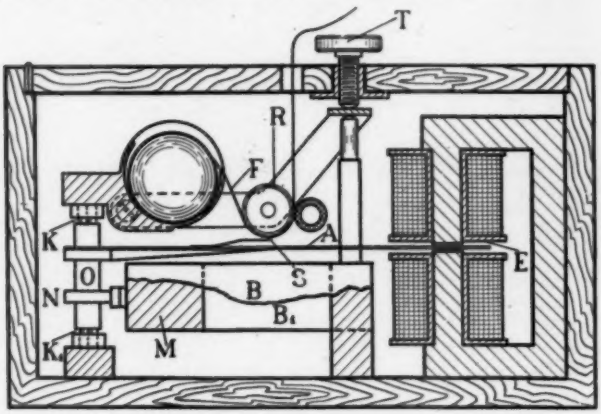


FIG. 12. *Diagrammatic plan of accelerometer (for vertical accelerations).*

ACCELEROMETER

This instrument has been designed for measuring the accelerations of motor vehicles, railway coaches, elevators, aeroplanes etc. It also pro-

vides a convenient means of studying the springing of vehicles and the effect produced on them by various road surfaces. It can also be used to compare the "comfort" of different types of tires when fitted to cars. The plan of the instrument when used for vertical accelerations is

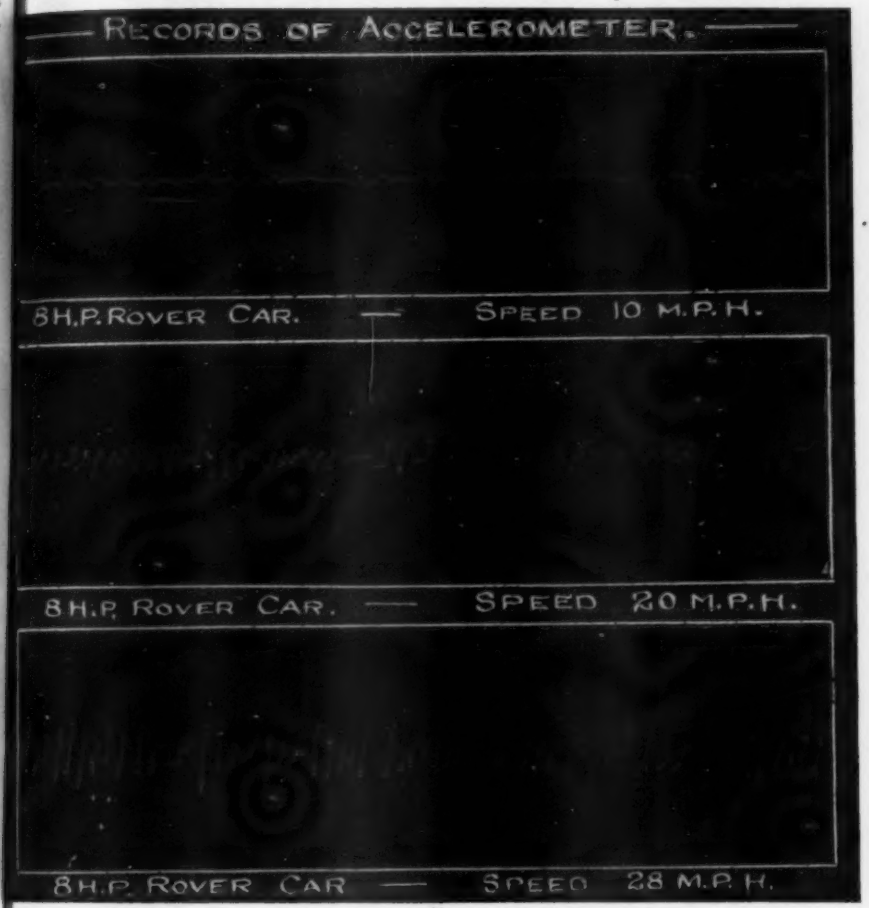


FIG. 13. *Accelerometer records.*

shown diagrammatically in Fig. 12. The mass M is supported by two flat steel strips B and B_1, and is connected by means of a steel strip N to a member O moving on knife-edges K and K_1. This member carries a flat strip of aluminum A, the extremity of which is free to

move between the poles of an electromagnet *E*. Fixed to the strip *A* is the stylus *S*. Any movement, therefore, of the mass *M* moves the stylus across the celluloid film *F* which is caused to pass around the roller *R* by means of a clock-work mechanism not shown in the diagram. The speed of the film may be varied from about 3 to 20 mm per second. The eddy currents set up by the movement of the aluminum strip between the poles of the electromagnet powerfully damp the movement,

FIG. 14. *Accelerometer records.*

and render the instrument almost critically damped. The natural period of the whole system is approximately 0.025 second. The pressure of the stylus on the film can be varied by means of the screw *T*. Fig. 13 shows three records taken on a small 8 hp. motor-car running at 10, 20 and 28 miles per hour respectively. The car was running over approximately the same piece of road and the record shows the increased vertical accelerations set up in the car by the higher speeds. The middle

record in Fig. 14 was obtained when the instrument was hung by an oscillating spring having a period of approximately 0.7 second. The lower record was obtained when the instrument was dropped out of a window from a height of about 22 ft. It will be observed that when first dropped from the window it oscillated slightly, and then recorded acceleration due to gravity. When the instrument reached the sheet in which it was caught, the acceleration upward was recorded. The upper record was taken on the running board of a 14 hp. motor-car. The same instrument can be used to measure either vertical or horizontal accelerations.

The details of the construction of all these instruments are due to Mr. W. G. Collins and Mr. W. J. Stallan.

MOULLIN TORSION METER

Many instruments have been designed to measure the torque on a shaft transmitting power. The generally accepted method is to measure optically the relative movement of two members fixed at a known distance from each other on the shaft. The angle made by these members relative to each other is observed at some part of the revolution and hence, knowing the speed of rotation of the shaft and its modulus of rigidity, the horsepower transmitted can be determined. In the instrument designed by Mr. E. B. Moullin of the Engineering Laboratory, Cambridge, England, the relative movement of these two members is measured electrically and continuously throughout the revolution, so that the instrument can be fixed in the ship's tunnel on the shaft, and the observations made at a distance. The Moullin torsion meter has been used to measure the torque transmitted on ships' shafts up to 10 inches in diameter, and transmitting 1500 hp. The instrument consists of an air-gap choker, one-half carried by a ring fixed to a point on the shaft, and the other half carried adjacent to the first but attached to a sleeve fixed to the shaft about 4 ft. away. The small diagrams[2] shown in Fig. 15 show the arrangement of the halves of the choker, of which the one *a* is fixed to the ring, and the other *b* is attached to the sleeve, the effective length over which the measurements are made being that between the normal planes of the two attachments. A small alternating-current generator supplies current to the windings *c* at 60 periods and about 100 volts. As the shaft twists, the gap opens for forward running (and closes in running astern) and the current increases in direct proportion to the gap, thus giving a scale in linear

[2] Reproduced from "Engineering," p. 764; June 13, 1924.

proportion so that the measurements on a record vary directly with the torque. Two chokers are fitted, one at each end of a diameter, so that they are in mechanical balance, and, being connected electrically in series, are unaffected by bending movements of the shaft. The arrangement forms in fact a compensation for side-way movement which would tend to open one air-gap and close the other by an equal amount. Current is led in and out of the chokers by two slip rings *d* and *e* mounted together, insulated from each other and from the shaft, and taking current from two brushes carried from any convenient fixed point. Correct measurement of torque depends on a definite length between the points of attachment of the two halves of the choker; this

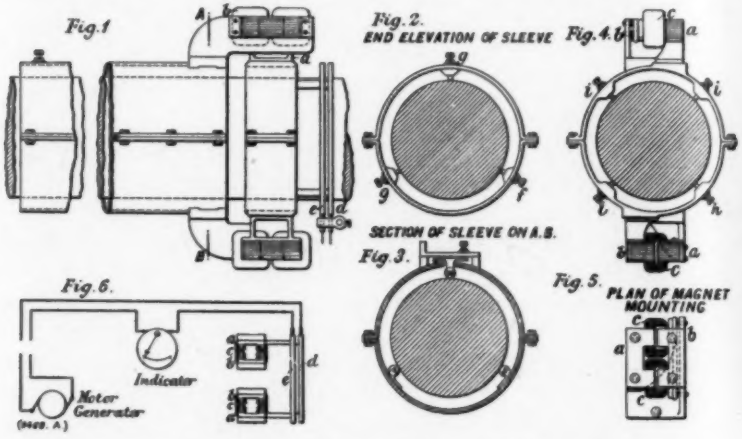


FIG. 15. *Diagrammatic sections of Moullin torsionmeter.*

is obtained in the case of the sleeve by means of a screw *f* with a conical point entering a short cylindrical hole in the shaft. At 120 degrees from either side of this pointed screw are two spherical-ended screws *gg*; the three screws taken together give a rigid fixing at this cross-section of the shaft. At the other end of the sleeve, in line with each of the three screws, are rollers, having their axes parallel to the axis of the shaft, so arranged near the ring as to support the end of the tube while allowing relative rotational movement between the sleeve and the shaft. Both the sleeve and ring are made in halves, the parts being bolted together.

The torsion meter is made in two forms, an indicating and a recording pattern. In its former pattern, designed for commercial work, direct

readings of the mean integrated torque are obtained on a suitably calibrated ammeter controlled by the gear. The indicator may be placed at any distance from the shaft and may be in multiple to give readings in various parts of the ship. Owing to the design of the small motor generator (0.2 kw), the readings are independent of changes in the voltage of the ship's continuous current supply used to drive this generator. For a permanent record the ammeter is replaced by a standard Duddell oscillograph, and by this means a continuous record is obtained of the instantaneous values in place of the mean reading given by the ammeter. Assuming that the current given by the generator has a frequency of 60, a measurement of the instantaneous torque is made 60 times per second. The time-scale on the record is about 15 cm per second. The vertical scale for the torque can be arranged to any convenient amplitude, and gives positive and negative readings for ahead and astern respectively.

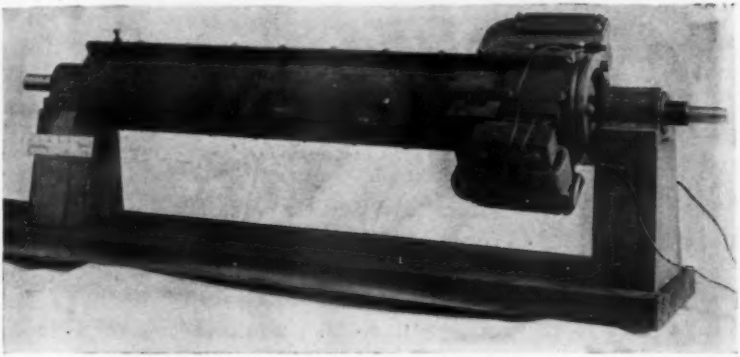


FIG 16 *Side view of Moullin torsionmeter.*

The calibration of the instrument is effected by mounting the coils in their working relative position on a screw-table giving readings to 0.01 mm; the relations between current and air-gap, when obtained and plotted, give a straight line over the working range. The relation between the air-gap and torque is calculated from the shaft dimensions, namely, its diameter and the length between the points of attachment of the torsion meter. The relation between torque and current being known, the ammeter can be calibrated to give direct readings. The coils are designed so that a change of 0.1 ampere is equal to a relative movement of 1 mm of the two halves of the choker. In assembling the gear on the shaft on board ship, the gap is adjusted until the ammeter

reading is the zero of the torque scale—which is not the zero of the ammeter scale, but is so chosen as to give room for recording astern readings.

FIG. 17. *End view of Moullin torsionmeter.*

Figs. 16, 17 and 18 show a torsion meter mounted on a shaft. One meter may be used to cover a large range of shaft diameters. In one case an apparatus constructed for measuring 400 hp, but temporarily fitted with a shaft much smaller than the 4-inch shaft intended for the full capacity of the instrument, enabled measurements to be made

FIG. 18. *Side view of Moullin torsionmeter.*

to 0.1 hp. with the same sleeve, rollers and chokers, and a disc mounted on the shaft (0.5 inch in diameter) as a roller path. The record reproduced in Fig. 19 was obtained at sea on a 10-inch shaft. It shows the torque from a 6-cylinder Diesel engine working on a 4-stroke

cycle. There are thus three peaks on the record per revolution of the shaft. The complete revolutions are indicated by means of a contact maker on the shaft, and are shown in the centre of the record. In this particular case the revolutions were 74 per minute. A curve drawn through the tops of the peaks gives the torsion curves during each separate revolution of the shaft. Fig. 20 is an interesting example of a record obtained of the forced oscillations of the shaft when the engine was slowing down, and shows how the variations of torque decrease as

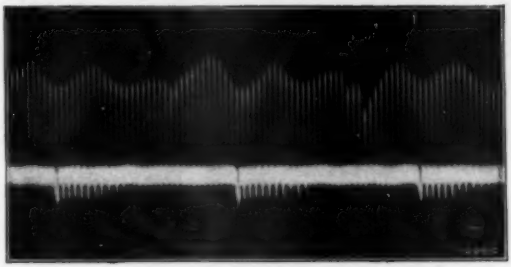


FIG. 19. *Record obtained with a Moullin torsionmeter showing the torque from a 6-cylinder Diesel engine working on a 4-stroke cycle.*

the speed gets further away from the neighborhood of resonance. The following values obtained on a steam trial show how useful and sensitive such a torsion meter can be. The actual values given in Table 1 are not the true values, but they have all been multiplied by a factor which renders them true for purposes of comparison, the values being obtained on a measured mile:—

TABLE 1. *Power and Speed.*

Trip	Horse power	Ship's speed over the ground in knots.
Up Clyde.....	1250	13.75
Down Clyde.....	1300	13.50
Up Clyde.....	1230	13.65
Down Clyde.....	1250	13.50

The hp readings were determined from the ammeter readings working on the torsion meter. It is interesting to note that after the ship had turned an increased speed was given, although the horsepower was slightly less, showing that either wind or tide was helping the vessel.

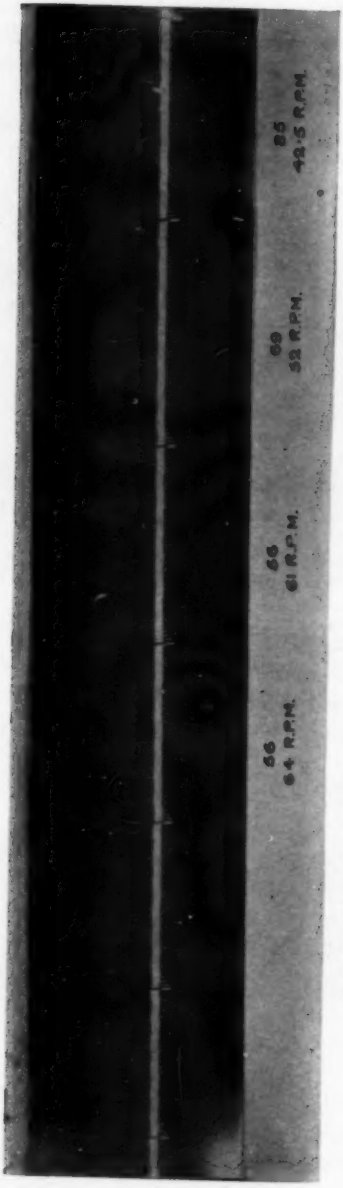
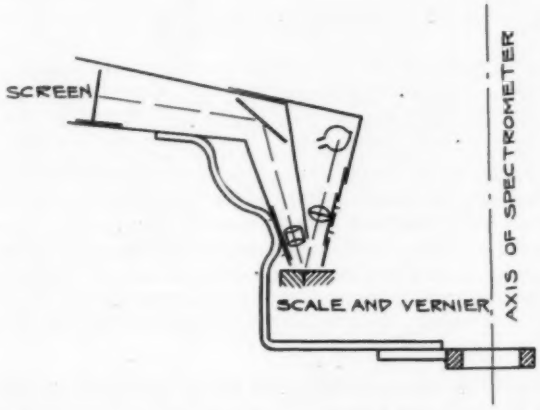


FIG. 20. *Torsionmeter Record showing forced oscillations of the shaft when the engine is slowing down.*

The operation of the oscillograph is not difficult, and as was previously explained, all this part of the apparatus can be mounted in an engine room or in an adjoining cabin, while the torsion meter itself remains unattended in the shaft tunnel.

CAMBRIDGE INSTRUMENT CO.,
45 GROSVENOR PLACE,
LONDON, S. W.

Scale Projection Device.—Spectrometer and other work where circular scales have to be read with accuracy becomes irksome when long series of readings are taken. This note describes a projection device which was constructed to facilitate the taking of readings when doing this class of work.

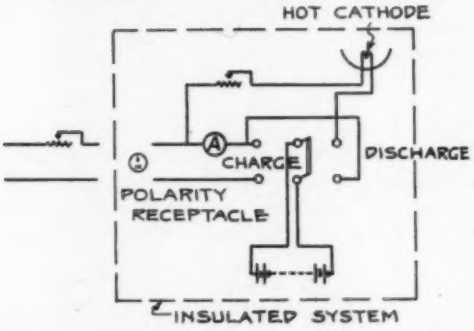


A low voltage concentrated filament automobile headlight lamp, two pocket flash light lenses, a two-lens magnifier, a front silvered mirror and a ground glass screen make up the optical system as is shown in the accompanying drawing. The image of the scale is inverted but in practice it was found to be of only slight disadvantage.

The containing case was mounted on the pedestal of the instrument so that it could be rotated about the axis and yet always have the scale focused on the screen. A foot switch control was placed in the lamp circuit since the lamp house was not designed for continuous burning. The scale and its vernier are projected with a magnification of about six diameters and are easily visible in a room with ordinary illumination.

ROY KEGERREIS, ANN ARBOR, MICHIGAN

Storage Battery Circuit.—The use of hot cathode apparatus requires a local heating circuit for which storage batteries are almost universally employed. The disadvantage in their use is that they must be frequently charged.



A wiring diagram which simplifies the necessary manipulations is shown in the accompanying figure. A single scale ammeter is made to read positive for both charge and discharge. Separate rheostats are provided for the two purposes since it is very frequently the case that high voltage and large currents are used for the charging of the batteries while the batteries themselves are of low voltage and fine current control is desired in the filament heating circuit. The connections are "fool proof" against burning out the filament during charging. The charging of the batteries is easily accomplished by inserting the polarity plug into its receptacle and closing the double-pole double-throw switch as is indicated. ROY KEGERREIS, ANN ARBOR, MICHIGAN

Comparison of Spectra Emitted from Core and from Aureole of an Arc in Sodium Vapor.—Forming an arc between a sodium cathode and an iron anode in an evacuated tube, Bartels observed that (until the sodium was nearly exhausted) it had a bright yellow core from which both principal and subordinate series were strongly emitted, and a deep green aureole from which the subordinate series lines were much more intensely emitted than the principal series. In addition there is a continuous spectrum extending from the limit of the subordinate series upwards toward higher frequencies, suggesting that an electron moving with finite speed through outer space may be drawn directly into one or the other of the $2p$ orbits, liberating in radiation an amount of energy equal to its initial kinetic energy plus the energy required to remove the valence-electron from the $2p$ orbit to infinity. The arc assumes an entirely different form when the sodium is nearly exhausted; the subordinate series are very feeble in its spectrum compared to the principal series. The theory is obscure. [H. Bartels; ZS. f. Phys. *25*, pp. 378-401; 1924.] KARL K. DARROW

PIEZO-ELECTRIC STANDARDS OF HIGH FREQUENCY

By W. G. Cady

ABSTRACT

It is shown that the qualities sought in a frequency standard are present to a high degree in properly mounted plates or rods of quartz, or steel rods excited by quartz plates, making use of the piezo-electric reactions of the quartz upon a resonant high-frequency circuit. The transverse effect (vibrations perpendicular to the impressed electric field) is chiefly utilized in the construction of piezo-electric resonators. Any given resonator may be excited to resonant vibration at its fundamental frequency or at any one of several overtone frequencies. The logarithmic decrement is very small, of the order of 0.001 or less. Hence a tube generator circuit of variable frequency may be tuned to a natural frequency of the resonator with very great precision.

The reactions of the resonator upon the circuit may be expressed in terms of characteristic changes in capacity and resistance. Curves illustrating this are shown, and it is seen that at resonance the resonator functions as a pure resistance.

In the construction of resonators, a quartz plate cut in the form of a comparatively narrow rod is placed loosely between two brass side-plates on an insulating base. Effects of changing temperature and of spacing of side-plates are discussed. For frequencies below 50 kilocycles per second steel rods are employed, excited to longitudinal vibration by having small quartz plates cemented to them at the center. By this construction frequencies as low as 3000 cycles have been reached. The upper limit in frequency, using very short quartz rods, is about three million cycles. Over this range the frequency of a vacuum tube generator may be synchronized with the resonator with a precision of one hundredth percent or better. Some of the various electric circuits that can be employed are briefly described.

The application of the resonator as a frequency stabilizer, and also as a device for generating electric oscillations, is explained. For these purposes either the transverse effect or the longitudinal effect (vibrations parallel to the electric field) may be used.

In the course of certain investigations on the properties and applications of piezo-electric crystals in 1917 and 1918, the writer became impressed with the extreme sharpness of resonance of plates of quartz and Rochelle salt when vibrating in alternating electric fields. This fact, together with the very pronounced reactions which the vibrating plates exerted on the electric circuit, suggested that suitably prepared piezo-electric plates might be made to serve as precise standards of high frequency, and as a means for generating electric oscillations of very constant frequency.

The chief qualities to be sought in a frequency standard are constancy with respect to time, temperature, and continued use; low decrement; ruggedness and simplicity; portability; and possibility of use without elaborate auxiliary apparatus. As will be seen, these qualities are present to a high degree in the piezo-electric resonator. Experiments extending over several years have proved that a resonator can

be constructed of quartz, or quartz and steel, which, depending upon its size, will serve as a standard for any frequency from about three thousand up to three million cycles per second. The range can undoubtedly be extended considerably in either direction. Throughout the range, wave-meters, tube generators, and other circuits of variable frequency can easily be synchronized with the resonators to an accuracy of one tenth percent, and with suitable care a precision better than one hundredth percent is attainable.

PRINCIPLE OF THE PIEZO-ELECTRIC RESONATOR

1. Owing to its unrivalled mechanical properties, quartz has thus far been found the most suitable material, though its piezo-electric constant is less than that of tourmaline, and far below that of Rochelle salt. On account of the occasional occurrence of twinning, and possibly other causes not yet recognized, not all quartz crystals are equally good. However, it is exceptional not to secure good plates from a crystal that appears flawless to the eye.

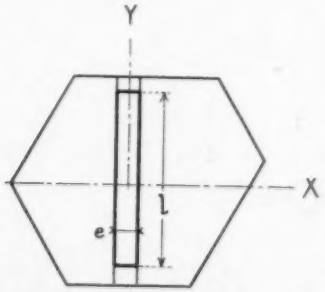


FIG. 1.—*Cross-section of a quartz crystal, in which the optical axis is perpendicular to the paper. X is one of the three electrical axes. The orientation of the piezo-electric plate is shown in heavy lines.*

The main facts concerning the piezo-electric action may be summarized as follows. Plates are cut from the crystal as shown in Fig. 1, having their length l perpendicular to one of the binary or electrical axes, their breadth b parallel to the optical axis, and their thickness e parallel to the electrical axis. When such a plate is placed between parallel plates of metal, so that the electric field is parallel to the e-direction of the quartz (that is, parallel to the electrical axis), an elongation in the l-direction and a contraction in the e-direction, or the reverse, tends to occur, according to the direction of the electric field. These two piezo-electric effects, in the l- and e-directions, are called the

transverse and *longitudinal* effects respectively. In general, the deformation of a crystal when in an electric field is known as the *converse* effect, while for historical reasons the term *direct* effect is applied to the generation of an electric field in a crystal when mechanically deformed. The reaction upon the electric circuit referred to in the first paragraph is due to the direct effect.

If the field between the condenser plates is alternating, the plate vibrates under the impressed piezo-electric stresses, with a frequency equal to that of the impressed field. This frequency may of course be that of one of the natural modes of vibration of the plate. The modes that are of chief importance in quartz are longitudinal vibrations, *i.e.* stationary compressional waves, in the *l*- or *e*-direction (Fig. 1). Any given plate possesses a fundamental frequency of vibration in each of these directions, together with overtones which are very nearly in harmonic relation to the fundamentals. Still other modes of vibration are practically always present and are sometimes an annoyance, particularly with the longitudinal effect (*e*-direction, Fig. 1). By making use of the transverse effect (longitudinal vibrations in the *l*-direction in narrow rods, as in Fig. 1), these troublesome nondescript modes of vibration can easily be eliminated, at least insofar as the fundamental and the first few overtones are concerned.

The relation between wavelength and frequency of longitudinal vibration in any direction in quartz may be calculated from the density and Young's modulus. The density of quartz is 2.654, and Young's modulus, Y, for extensions parallel to either the X or the Y axis is about 8×10^{11}. Hence for the velocity of compressional waves for either the longitudinal or the transverse effect we have $v=\sqrt{Y/\rho}=5.5\times10^{5}$ cm/sec. From this we find the wavelength λ in the crystal to be roughly

$$\lambda = 5500/f\,, \qquad (1)$$

where f is the frequency in kilocycles per second, and λ is in millimeters. Owing to the effect of cross section upon frequency and to other causes, this value differs somewhat for different plates. If the boundary surfaces of the plate are free, the fundamental vibration has a node at the center of the plate, so that l is a half wavelength for the transverse effect, and e a half wavelength for the longitudinal effect. This freedom of the boundary surfaces is secured by the method of mounting described below.

For the fundamental frequency we have, on substituting $2e$ or $2l$ for λ in Eq. 1, approximately $f=2750/l$ for the transverse effect, and

$f = 2750/e$ for the longitudinal effect, f being in kilocycles per second and l and e the length and thickness of the plate in millimeters.

It is sometimes convenient to express l and e in terms of the wavelength of electromagnetic waves in air. From the equations above it is seen that the dimension of the crystal in millimeters, measured in the direction of vibration, is slightly less than one hundredth of the wavelength in meters, since the latter equals $3\times10^8/f$. In other words, about 110 meters electromagnetic wavelength may be allowed for each millimeter of the crystal in the direction in which vibrations of the type described above occur. Thus, a plate 20 cm long and 1 mm thick will resonate longitudinally (with respect to the electric field) under an impressed wavelength of about 110 meters, and transversely when the impressed wavelength is about 2200 meters. The excitation of overtones will be further discussed in Section 3.

In the case of quartz plates vibrating in the e-direction (longitudinal effect), there is usually found, in addition to the fundamental frequency mentioned above, corresponding to about 110 meters electromagnetic wavelength for each millimeter of thickness of plate, a second prominent mode of vibration of somewhat lower frequency. This mode corresponds to a wavelength of about 150 meters per millimeter. Many plates can be made to function as piezo-electric oscillators at this latter frequency (see Section 11).

2. In previous papers the writer has given the theory of the piezo-electric resonator.[1] The term "resonator" is applied to the device when connected in a tuned oscillating circuit as a frequency standard or as a stabilizer (Section 9). When used with an electron tube amplifier for the purpose of generating oscillations it is called a "piezo-electric oscillator."

It is only necessary to state here that the reactions of the resonator upon a high frequency circuit are fully accounted for, and that they consist in characteristic changes in the effective capacity and resistance of the resonator as the frequency is varied in the neighborhood of resonance. As a typical instance, a quartz resonator approximately $30\times4\times1.4$ mm may be mentioned, which when vibrating longitudinally (transverse effect) has a fundamental frequency of 89870 cycles per second. As this resonant frequency is approached, the electrostatic capacity of the resonator, which normally is 4.5 micromicrofarads, increases until at 89860 cycles the equivalent parallel capacity is

[1] Proc. Inst. Radio Eng. *10*, p. 83; 1922; Phys. Rev. *19*, p. 1; 1922.

42.2 $\mu\mu f$. The capacity then decreases quickly to the normal value at resonance, passes through zero and becomes negative, reaching the value of $-32.2\ \mu\mu f$. at about 89880 cycles per second. If the frequency is further increased, the capacity gradually returns to the normal value again.

These variations in equivalent parallel capacity are represented in Fig. 2, in which the horizontal broken line indicates the "normal" capacity of the resonator. Fig. 2 shows also the accompanying changes

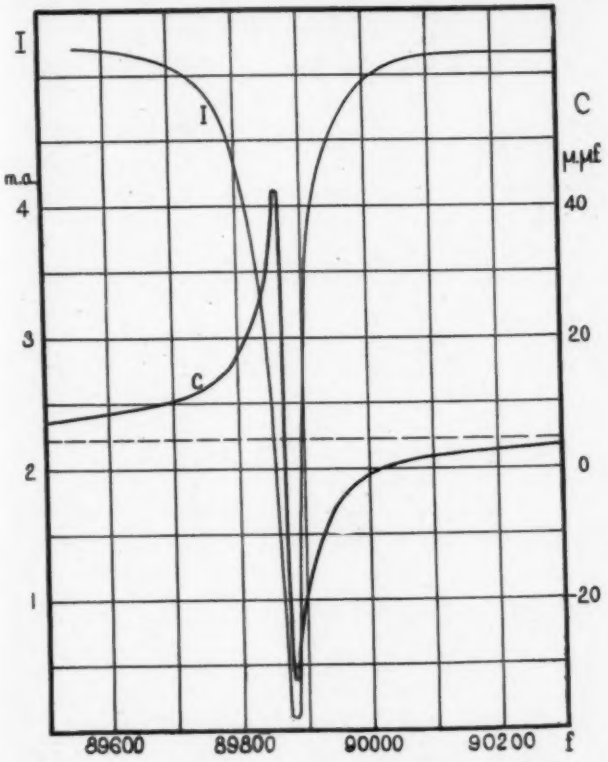


FIG. 2.—*Characteristic curves of a piezo-electric resonator 30.7 mm long.*

in I, the current in milliamperes in the oscillating circuit in which the resonator is connected, as the frequency is varied. For a more detailed account of these observations the paper in Proc. Inst. Radio Eng. referred to above should be consulted. As is seen in Fig. 2, the current descends to a very sharp minimum at resonance, this value being

smaller the lower the resistance of the electric circuit. The equivalent parallel resistance of this resonator is practically infinite except close to the resonant frequency, when it descends to a sharp minimum of about 20000 ohms. A variation in frequency of one tenth percent from the resonant value causes the equivalent parallel resistance to increase more than a hundred fold. The form of the curve of resistance is similar to that of the current curve in Fig. 2.

From the data above the capacity reactance at resonance is readily found to be about 400,000 ohms. Hence at resonance this resonator functions practically as a *pure resistance*. The logarithmic decrement of the resonator is only 0.00066 per cycle. Quartz plates of other sizes show similar characteristic effects at resonance.

The effectiveness of the resonator as a frequency standard is due entirely to these very pronounced and rapid changes in capacity and resistance over a range of frequency comprising a small number of cycles in the neighborhood of resonance.

Extreme care in squaring the ends of plates and in grinding opposite faces exactly parallel is unnecessary, but there seems to be a slight advantage in polishing the surfaces.

MOUNTING OF QUARTZ PLATES FOR USE AS RESONATORS AND OSCILLATORS

3. It was hoped that in the construction of resonators for frequencies above 1000 kilocycles per second, where the length of the quartz "rod" used in the transverse effect begins to be inconveniently small, resonant vibrations of broad plates in a direction perpendicular to their flat faces could be employed, using the longitudinal effect. By the use of thin plates, resonance can be obtained at very high frequencies. Unfortunately, broad plates of quartz when vibrating in this way possess a multitude of resonant frequencies in the neighborhood of the fundamental—indeed, it is usually impossible to say which frequency *is* the fundamental. For this reason it has been found best to utilize only the transverse effect in the construction of all resonators, even for highest frequencies.

MOUNTING OF QUARTZ RODS, TRANSVERSE EFFECT

For most purposes the best method of mounting has been found to be as shown in Fig. 3. *A* is a small block of bakelite or other insulating material, across the top of which a wide channel is cut. In this channel are firmly pinned two brass pieces *BB* to which short rods *CC* are attached, to serve as terminals for connection to the electric circuit.

The quartz plate, or rod, *Q*, stands on edge in the pocket between the two plates *BB*, which form the "coatings" of the resonator. *D* is a cover of the same material as the base, attached to the base by four screws. It must of course not press upon the quartz. A clearance of one or two

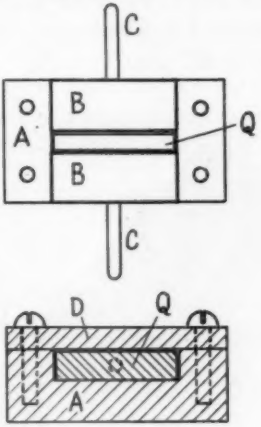


FIG. 3.—*Mounting of quartz rod for excitation of fundamental frequency. Above: top view. Below: cross section through center.*

tenths of a millimeter is left all around the quartz to insure freedom to vibrate. There is a slight advantage in having the bottom of the pocket slightly crowned, in order to allow the ends of the quartz rod to vibrate

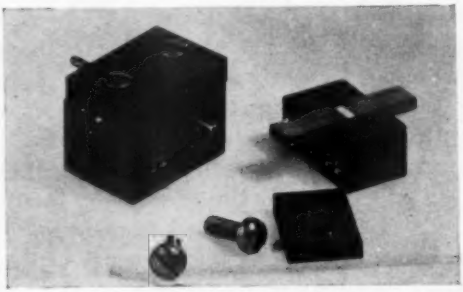


FIG. 4.—*Two small resonators, for frequencies of 757 and 860 kilocycles per second.*

more freely; but this precaution is not so essential as might be supposed, owing to the extremely minute amplitude of vibration, which is generally of the order of 0.001 mm or less.

A photograph of two small resonators is shown in Fig. 4. The cover of the resonator at the right has been removed, showing the quartz rod lying between its two brass plates. This rod is 3.2 mm long, and resonates at a frequency of about 860 kilocycles per second.

Fig. 5 shows four resonators in a common mounting. The cover in this case is of glass, through which the quartz rods may be seen. Their lengths range from 1.76 to 30.3 mm.

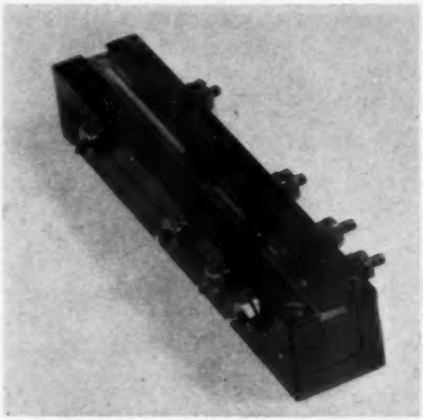


FIG. 5.—*Four quartz resonators in a common mounting with glass cover. The fundamental frequencies are 91.66, 235.9, 762.0 and 1524 kilocycles per second.*

A resonator mounted as in Fig. 3, with coatings of the same length as the crystal, will respond strongly at the frequency of its fundamental vibration, but only very weakly at overtone frequencies. In order to excite the latter effectively, the metallic coatings are made very narrow, for example covering only one seventh of the length of the crystal, the remainder of the sides of the pocket consisting of insulating material, as shown in Fig. 6. In this case the fundamental and all overtones up to that having about seven times the frequency of the fundamental can be equally well excited.

4. The fundamental and overtone frequencies of a quartz plate vary to a slight extent with the width of the air space and with the extent to which the plate projects beyond the coatings. In order to understand this, one may consider the analogy between the polarized quartz plate and a short, thick permanent magnet. When the coatings are close to

the quartz, they and the electric circuit are analogous to a massive yoke of high permeability, allowing the induction to attain a large value. As the air gap increases, the plate partially depolarizes itself, owing to the turning back of some of the lines of electrostatic induction that are produced by its state of strain. The piezo-electric action of this depolarizing component is always such as to tend to make the plate regain its normal form. That is, the effective value of Young's modulus is larger, the greater the air gaps. In the most extreme case hitherto observed, the increase in frequency when the coatings, originally touching the quartz, were removed to a great distance, amounted to

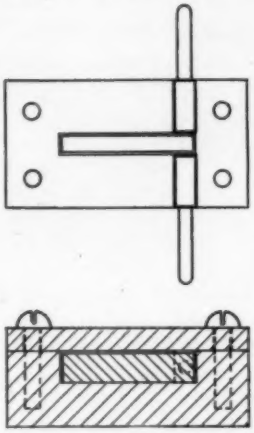


FIG. 6.—*Mounting of quartz rod for excitation of fundamental overtones.*

about six tenths percent. Even when the metallic coatings are fixed, as in Fig. 3, a shift in the position of the quartz from the center of the pocket to contact with one of the coatings may produce a change in frequency of a few hundredths of a percent. In cases where this is objectionable, means are easily provided for holding the plate in the center of the pocket; or the air gaps may be reduced as nearly as possible to zero. In practice the air gaps are, indeed, made as small as is consistent with freedom of vibration of the quartz, and this has the added advantage of insuring the strongest possible reaction of the resonator upon the circuit.

QUARTZ-STEEL RESONATORS

5. The writer has used as resonators quartz plates or rods of lengths from 1.35 mm up to 120 mm, the frequencies varying from 2000 to 23

kilocycles per second. Owing to the comparatively high cost of very long quartz plates, it has been found expedient for all frequencies below 50 kilocycles to use as resonators flat steel rods excited by small plates of quartz cemented to their sides at the middle. Such a resonator is shown in Fig. 7. Good results have been obtained with steel rods about 9×3 mm in cross section. The quartz plates are 1 to 2 cm long, a millimeter thick, and are cemented to the steel with solid shellac. The steel itself serves as one "coating" for each plate, the other coatings being of tinfoil connected in parallel as shown in Fig. 7. The same polarity of each plate must of course face the steel. A small hook is attached to the steel rod at its center, between the quartz plates, to serve as a suspension and as one of the terminals of the resonator, leaving the rod free to vibrate at one of its natural frequencies. Through the action of the transverse effect, the quartz plates expand and contract longitudinally when connection is made to a source of e.m.f. of the right

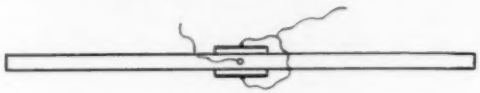


FIG. 7.—*Quartz-steel resonator, the exciting quartz plates having tinfoil coatings electrically connected. The hook between the quartz plates, by which the steel rod is suspended, serves as the other terminal.*

frequency, causing a periodic condensation and rarefaction at the center of the steel rod, whereby longitudinal vibrations are excited in the rod.

The resonant frequency is essentially that of the steel rod, and electric reactions occur exactly as with the quartz resonator, with an intensity sufficient to produce a strong response. The decrement is not very different from that of the larger quartz resonators. In addition to the fundamental, a number of overtones can be excited.

6. Fig. 8 shows two quartz-steel resonators, mounted, of frequencies 28.94 and 14.43 kilocycles, the lengths of the rods being 90.5 and 180 mm respectively. The two resonators shown in this figure, as well as those in Fig. 5, were compared with standard wave-meters in the United States, Italy, France, and England in 1923.[2]

The writer has constructed resonators similar to those shown in Fig. 8 over a meter in length, which emit an audible note when excited.

[2] Proceedings of the Institute of Radio Engineers, *12*, p. 805, 1924.

They may therefore be used as high-pitch standards of sound, and are well suited for experiments on resonating tubes etc.[3]

In the case of these long rods, by cementing to the steel, instead of one pair of quartz plates, two pairs of plates cut from Rochelle salt crystals, the plates being connected to the input and output circuits of an electron tube amplifier, the vibrations are made self-sustaining. No tuned electric circuit is necessary.[4] This method is explained in Section 10 below.

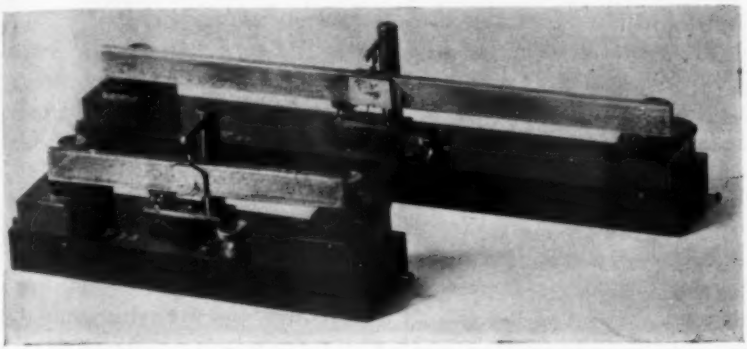


FIG. 8.—*Two quartz-steel resonators, the rods being suspended, ready for use. For transportation the rods are unhooked and clamped to the base. Each instrument has a wooden cover.*

TEMPERATURE COEFFICIENT OF RESONATORS

7. As is to be expected, the temperature coefficient of frequency under ordinary conditions is negative, *i.e.* the frequency decreases with rising temperature. For the transverse effect in quartz rods the coefficient amounts to about five parts in a million per degree C.[5] Quartz plates vibrating under the longitudinal effect in a direction normal to their faces usually have a coefficient of about twenty parts in a million. The coefficient for quartz-steel resonators is about 100 parts in a million.

For most purposes, so far as ordinary variations in room temperature are concerned, the effect of temperature upon frequency may be neglected.

[3] Phys. Rev. *23*, p. 558, 1924 (abstract).

[4] Phys. Rev. *21*, p. 371; 1923. In this abstract the production of a continuous note from a tuning fork by means of Rochelle salt plates is also described.

[5] Powers, Phys. Rev. *23*, p. 783; 1924 (abstract).

APPLICATION OF THE RESONATOR AS A STANDARD OF HIGH FREQUENCY

8. Inasmuch as the resonator is itself a minute condenser, it is normally connected in parallel with the tuning capacity of a high-frequency circuit. This may be either the generator circuit or a secondary circuit coupled thereto. In either case care must be taken not to let the amplitude of vibration become so great as to fracture the crystal. The most suitable type of generator is the electron tube. Results have been obtained with resonators connected to buzzer-driven circuits, but usually with some difficulty.

The response when the generator is gradually tuned until its frequency synchronizes with a natural frequency of the resonator may be either visual (ammeter or galvanometer) or audible (telephone receiver or loud speaker). The visual method can be made to yield somewhat more precise settings.

Only a brief resumé of the chief methods of comparing the resonator with a high-frequency circuit is in place here.[6]

Any generator circuit may be calibrated with a precision of 0.1% by connecting a number of resonators of various known frequencies in succession across its condenser terminals. By using resonators of the type shown in Fig. 6, overtones as well as the fundamental may be employed as standards. In each case the capacity of the condenser is varied back and forth, and the precise setting noted at which a click or "chirp" is heard in the telephone. This sound is due to the fact that the resonator, when once set into vibration, of itself generates a high-frequency electromotive force and produces beats with the high-frequency current already present. The reaction of the crystal upon the circuit is so pronounced that clicks can usually be heard with one terminal entirely disconnected. Sometimes it suffices *merely to place the resonator in the electric field* of a condenser or coil, without any further connection to the circuit: the crystal absorbs and re-radiates sufficient energy even in this case to cause a faint sound to be heard on passing through the resonant frequency. If a telephone is not used, resonance may be indicated by a sudden dip in the reading of a meter in the grid or anode circuit of the oscillating tube.

A generating circuit that has thus been precisely adjusted to resonator frequency may be used to calibrate a wavemeter or other tuned circuit; or it can serve as a source of accurately known frequency in high-

[6] For further details and modifications see Cady, Proc. Inst. Radio Eng. *10*, p. 109; 1922, and *12*, p. 805; 1924.

frequency measurements of capacity etc., or as a master oscillator for radio transmission.

It is often advantageous to place the resonator and indicating device in a secondary circuit loosely coupled to the generator circuit. In other respects the method is like that described above. If the indicating device in the secondary circuit is a thermo-element connected to a galvanometer of high sensitivity, extremely precise settings of the generator frequency are possible. As resonance is approached, the crystal absorbs energy, causing the current in the secondary circuit to fall to an extraordinarily sharp minimum at resonance, as already depicted in Fig. 2. Even at frequencies as high as 1500 kilocycles per second, settings can be made with a precision of a few parts in a hundred thousand.

APPLICATION OF PIEZO-ELECTRIC CRYSTALS AS FREQUENCY STABILIZERS AND OSCILLATORS

9. To a greater or less extent a resonator of any of the types described above will stabilize the frequency of the generator circuit to which it is connected. The crystal acts merely as a small additional capacity except close to its own natural vibration frequency, when, according to the principles outlined in Section 2, the effective capacity of the crystal changes with the frequency in such a manner as to tend to compensate for the variation in frequency. Thus any disturbance which without the resonator would produce a change of one percent in frequency, may cause a change only one thirtieth as great when the resonator is connected.[7]

For good stabilization, quartz plates should be considerably larger than is necessary for resonators when used as frequency standards. At the higher frequencies, up to 3000 kilocycles per second, the longitudinal effect is employed, using thin plates with an area of several square centimeters. A flat metal plate serves as the "coating" on each side of the quartz, which may lie either horizontally or be stood on edge vertically. Such plates often stabilize well at any one of a number of different frequencies.

The chief difficulty with this method of stabilization lies in the fact that a disturbance in the circuit of more than a certain critical amount will cause the frequency to jump abruptly to a value so far from that of the resonator that the latter ceases to vibrate.

[7] Proc. Inst. Radio Eng. *10*, p. 112; 1922.

10. A quartz plate that stabilizes well can in general be made to serve also as a *piezo-electric oscillator*. When properly connected to the electron tube, a sufficiently large fraction of the power which it receives from the tube and which maintains it in vibration, is returned to the grid to sustain the electric oscillations in the circuit, without any electrical tuning or auxiliary feedback whatever. Of the various possible modes of connection, only two need be mentioned here.[8]

The first method requires a quartz plate provided with two pairs of coatings, connected to the output and input respectively of an amplifier tube, as in Fig. 9. Any slight change in the potential of the anode of the tube will vary the electric field between the coatings *B* and thus set the quartz plate into vibration. These vibrations, through the operation of the "direct" piezo-electric effect (Section 1), will induce charges

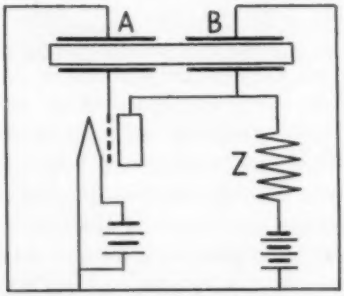


FIG. 9.—*Quartz plate with two pairs of coatings A, B, connected to an amplifying tube, generating oscillations with mechanically tuned feedback.*

on the coatings *A* and thus vary the potential difference between grid and filament. The vibrations are maintained through the amplifying action of the tube, but the frequency is determined by the mechanical vibrations of the quartz plate. The device is somewhat analogous to the tube-driven tuning fork.[9] Thus far this method has been used successfully only with the transverse effect, at frequencies from about 200 kilocycles down. An advantage in this method of generating oscillations lies in the fact that the same narrow rods used as frequency standards can be employed here, only provided with two pairs of coatings instead of one. Best results are generally obtained by using a two- or three-stage amplifier instead of the single tube shown in Fig. 9. The value of the impedance in the output circuit is practically without

[8] Cady. Proc. Inst. Radio Eng. *10*, p. 111; 1922.

[9] Eckhardt, J. Franklin Inst. *194*, p. 60; 1922; Eccles & Jordan, Electrician 82, p. 704, 1919; Ferguson, Bell Syst. Tech. J. *3*, p. 145; 1924.

effect upon the frequency. Thus Z in Fig. 9 may be either a resistance or an inductance. The oscillations are of maximum intensity when the output circuit contains a coil and condenser in parallel, tuned to crystal frequency. This tuning affects *only* the intensity, however, and is without appreciable effect upon the frequency, which is determined solely by the mechanical vibrations of the quartz rod.

The power output is considerably less than a watt. This method of generating oscillations of very constant frequency should prove useful in making electric measurements at high frequency, or in other cases where low-power oscillations of constant frequency are desired. As was stated in Section 6, this is the method employed by the writer for driving tuning forks and long rods at audio frequencies.

11. The second type of piezo-electric oscillator requires a quartz plate with only one pair of coatings, but the plate must be much larger in extent than is needed in the case of resonators for frequency standards. A good plate will usually oscillate at either one of two widely different frequencies, the higher one according to the longitudinal effect, depending upon its thickness, and the lower according to the transverse effect, depending upon the length of the plate in a direction perpendicular to the electrical and optical axes (*i.e.* the l-direction, Fig. 1). One of the coatings of the quartz plate is connected to the grid of an electron tube, the other to either the filament or the anode. The plate is driven by energy derived from the tube, and by its reactions impresses periodic potential variations upon the grid. The impedance of the output circuit of the tube must be adjusted approximately to the particular frequency of vibration that it is desired to excite in the quartz plate. This type of piezo-electric oscillator was first described by Pierce.[10] It seems at present to be the most promising piezo-electric device for radio transmission, and it furnishes also an excellent source of constant high frequency oscillations for laboratory use, especially on account of the large number of harmonic frequencies that are present in the output circuit.

WESLEYAN UNIVERSITY,
MIDDLETOWN, CONNECTICUT,
NOVEMBER 5, 1924.

[10] Pierce, Proc. Am. Acad. Arts and Sci. 59, p. 81; 1923. See also Shaw, Q. S. T. 7, p. 30 1924.

Pulling Electrons out of a Metal.—Experiments on a contact formed by laying a metal plate upon the surface of a block of poorly conducting substance (such as agate or lithographic stone) show that the current driven across it by an emf of given magnitude is greater when the emf is so directed as to drive electrons out of the metal than when it is oppositely directed. Kramer thinks that this is due to the intense potential-gradient across the interspace (or film, or layer, or whatever it may be called) between the metal and the stone, which, owing to the small separation of the two (estimated both from the electrostatic attraction between the plates, and from the capacity of the system, as being 1 to 30μ) may amount to a million volts per cm. This is comparable with the fields which suffice to draw electrons out of cold metals into vacuum. The effect is most pronounced when the metal surface is roughened, which seems reasonable. The ratio of the currents driven in the two senses by equal voltages has a maximum at a certain value of voltage; up to this point it is supposed that the field gradient is increasing as the interspace is narrowed, beyond this point it is supposed that the interspace is bridged by conducting connections which diminish the P. D. and the field gradient; these of course are *ad hoc* assumptions. [W. Kramer, Stuttgart; ZS. f. Phys. *27*, pp. 74-82; 1924.]

KARL K. DARROW

Spectra of Al, V, Ti, and Sc.—These exceedingly complicated spectra are searched for multiplets, and systems of multiple levels for the valence electron are then arranged so as to account for the lines composing these multiplets. Particular attention is paid to the lines occurring in the absorption spectra of the heated vapors, for the level (or levels) from which electrons are extracted when these frequencies are absorbed is the level (or levels) which the valence electron usually occupies in the normal atom. In the absorption spectrum of Al the two subordinate series appear, indicating that the level normally occupied by the valence electron is the $2p_2$ level. The spectrum of V is analyzed into sextets and quadruplets, the spectrum of Ti into quintets and triplets, the spectrum of Sc into doublets and quartets. From the absorption lines it is concluded that the deepest lying level of the V atom is a *F* or quadruple level, hence presumably to be designated as 4_4 in Bohr's notation; the deepest lying level of the Ti atom is a *F* or triple level, also to be designated as 4_4; the evidence for Sc is still insufficient. It is not possible to extrapolate to the limit of any series with sufficient accuracy to determine the absolute values of the terms. The experimental material for this analysis consists partly of data from Mt. Wilson, partly of data from the Bureau of Standards, and partly of photographs by the authors themselves of the absorption spectra of heated vapors of the four specified metals. [H. Gieseler and O. Grotrian, Potsdam Astrophysical Observatory; ZS. f. Phys. *25*, pp. 342-366; 1924.]

KARL K. DARROW

A UNIVERSAL POLARIMETER*

By C. A. Skinner

I. The Instrument

1. INTRODUCTORY

Polarized light is treated in general as an elliptic vibration of which rectilinear and circular vibrations are merely special cases. To completely define its character six factors are necessary: direction of propagation, wave-length, brightness—three factors which do not distinguish it from natural light; direction of rotation of the light vector, ratio of the minor to the major axis of the elliptical path, and the azimuth of, say, the major axis—three factors special to polarized light. In any discussion of polarized light as such, the first three factors may be neglected and the other three reduced to two, namely: *ellipticity* and *azimuth*. The ellipticity may be defined as the ratio of the minor to the major axis, positive in sign for a counter clockwise rotation of the light vector when viewing toward the light source and negative for the opposite direction of rotation. The azimuth may be defined as the smaller positive angle which the major axis makes with a fixed direction of reference, when looking toward the source. Circularly polarized light is characterized by an ellipticity equal to ± 1 and plane polarized light by zero ellipticity. The first has an indeterminate and the second a definite positive azimuth.

The general form of vibration being elliptical, any instrument which measures with precision both ellipticity and azimuth may logically be called a polarimeter. This name is commonly given however to the instrument specially designed for measuring simply the azimuth of plane polarized light, while the less common instrument which measures ellipticity has been designated as an elliptic analyzer. With the increasing use of the latter type it seems desirable to have the names of the two instruments suggest their close relationship, and as well, the distinction between them. Designating the one as an *azimuth polarimeter* and the other an *elliptic polarimeter* is intended to serve this purpose. The instrument described here being readily adaptable to either or to both types of measurements it is designated a *universal polarimeter*.

* Published by permission of the Director of the Bureau of Standards of the U. S. Department of Commerce.

2. METHODS AVAILABLE

For monochromatic analysis so-called sensitive tint methods are obviously inapplicable. This leaves, for precision measurements, a choice between the method of measuring the shift of extinction bands and that which makes a brightness match between two parts of the field of view. The latter depends for its accuracy on the ability of the observer to set two or more contiguous parts of the field of view for a brightness match. It is generally conceded that where both are applicable, the half-shade method is, for precision measurements, to be preferred.

In the design of half-shade systems two important factors must be borne in mind. The accuracy of setting is increased by having the brightness of one of the matched parts diminish as that of the other augments, and by having a vanishing dividing line between the contiguous fields.

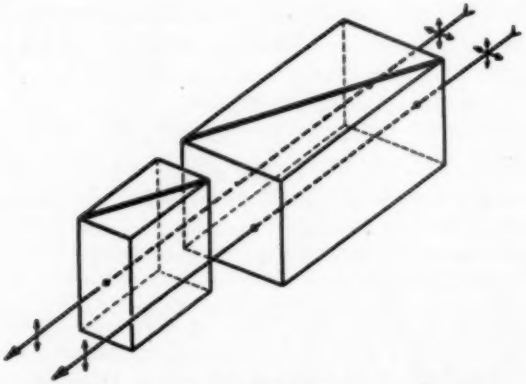


FIG. 1. *Lippich half-shade polarizer. Field brightness, as seen through the analyzing nicol, is increased by rotating whole nicol to a larger azimuth relative to the half nicol. (Used as analyzer the direction of the light is reversed).*

Azimuth Half-Shades.—Various devices are employed as half-shade systems in azimuth polarimeters for measuring rotation of plane polarized light. One of the simplest and most common consists in having a polarizer furnish plane polarized light of slightly different azimuths in the contiguous parts of the field. For this purpose the Lippich half nicol (Fig. 1) is pre-eminently the best—consisting of a regular polarizer covering the entire field, followed by the "half nicol" at a slightly different azimuth covering part of the field. In setting, the

analyzing nicol is adjusted for a brightness match at the extinction azimuth. A distinct advantage possessed by this system is its variable sensibility—the matched field being raised, when desired, to a higher brightness level, though lower setting accuracy, by rotating the whole nicol to a larger azimuth with respect to the half nicol. For some work it has a marked disadvantage arising from the unequal transmission of the half-shade parts, which entails an unsymmetrical setting of the analyzing nicol. In measurements on plane polarized light this effect is readily eliminated but with ellipticity present important difficulties arise. As a consequence the non-adjustable Cornu-Jellett "split" nicol (Fig. 2) must be accorded the same place in ellipticity measurements that the Lippich holds in plane polarized light measurements. Furnishing as it does equal transmission in the matched parts of the field the effects are equal and symmetrically introduced.

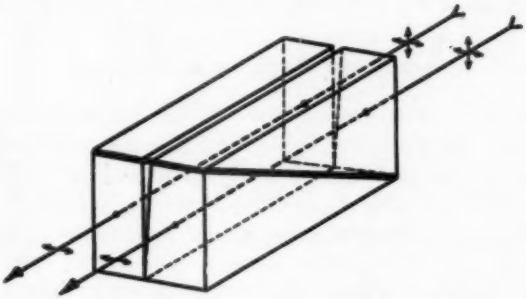


FIG. 2. *Cornu-Jellett half-shade nicol showing section removed after which the parts are closed together giving slightly different azimuths of vibration in the two parts of the field.*

Compensators.—Ellipticity measurements are always made by introducing in the given elliptically polarized beam a bi-refringent plate ("compensator") which "restores" the light to either a rectilinear or a small definite elliptical vibration. Common among such are the Soleil-Babinet variable thickness compensator (Fig. 3), the stressed glass compensator of variable stress and the simple bi-refringent plate, of mica for example, which depends on a variable azimuth for its range of compensation.

Because of its relative ease of construction and the convenience of application, the variable azimuth compensator has been chosen here in spite of the fact that any single plate is more restricted in its range of measurement than the variable order type. For example, although a quarter-wave plate will completely restore any elliptical vibration,

dispersion of the bi-refringence makes a quarter-wave plate for any given wave-length effectively less than a quarter-wave for all others.

Elliptic Half-Shades.—Among elliptic half-shades the Bravais bi-plate (Fig. 4) is perhaps the best known. This consists of two quarter-wave plates with their edges in juxtaposition and their respective

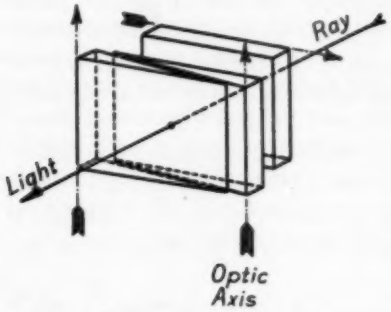


FIG. 3. *Soleil-Babinet variable order compensator.*

azimuths at 45° to this dividing line—90° to each other. When placed between two crossed nicols the two parts are matched at all azimuths of the half-shade—the brightness level of the field alone varying with the azimuth. The match is destroyed however as soon as the incident vibration becomes elliptic in form; hence the bi-plate serves to detect

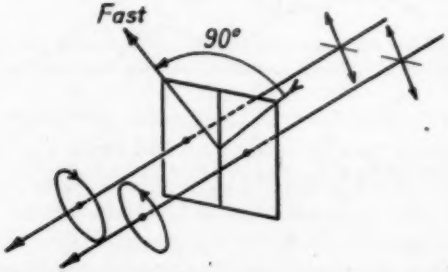


FIG. 4. *Bravais bi-plate—an elliptic half-shade in which the two parts are of equal and opposite order (quarter-wave). Working azimuth with respect to the nicol should be small to avoid excessive brightness levels.*

when elliptically polarized light has been completely restored. The optical symmetry of the Bravais bi-plate would make it an ideal half-shade were it not for the impossibility of obtaining a satisfactory dividing line; for, by varying its azimuth with respect to that of the

analyzing nicol the brightness level of the field can be varied without shifting the "zero" position.

The Brace half-shade which consists of a very thin sheet of mica imbedded in Canada balsam and covering half the field gives a practically vanishing dividing line. It also can be used at various brightness levels, but commonly is set at the position of maximum effect.[1] When using this half-shade the light is not completely restored to a rectilinear vibration but to a definite small ellipticity instead.

Tool's combination of an azimuth and an elliptic half-shade,[2] in which a Brace elliptic half-shade is attached to a Cornu-Jellett half-shade nicol gives undoubtedly the most precise method yet developed for measuring both azimuth and ellipticity of a given vibration.

The instrument described here is readily adaptable, as the occasion demands, for use as a simple azimuth polarimeter, a Brace elliptic polarimeter, or the more elaborate Tool general polarimeter to which the design especially conforms.

3. DESCRIPTION OF THE INSTRUMENT[3]

The principle of the Stokes' instrument,[4] after which Tool's modification is patterned, is illustrated by the optical train of Fig. 5. In the Stokes instrument the compensator is carried by a main graduated circle, the azimuth of which is referred to a fixed direction, and the analyzing nicol is mounted on a second graduated circle carried by the main circle, and its azimuth measured relative thereto. The operation consists in adjusting compensator and nicol for minimum field brightness at two complementary positions and noting the change in both circle readings which results therefrom.

Experience shows that the observer can interpret his results more readily by having the nicol settings as well as compensator settings referred to a fixed direction. We have therefore mounted compensator and nicol on independent circles which are read with respect to the same fixed direction. The instrument is shown in Figs. 6 and 6*a*, having been

[1] A stressed glass plate is sometimes employed as a substitute for the Brace mica strip and some exceptional claims made in regard to its sensibility. The mechanical difficulties attendant upon the production of a uniform field and vanishing dividing line are, relative to the mica strip, insuperable, so that it is difficult to conceive that the stressed plate should be preferred by any observer who has had experience with a Brace half-shade of variable azimuth.

[2] Tool, A. Q., Phys. Rev., *31*, p. 1; 1910.

[3] Instrument makers will be furnished, on application to the Bureau of Standards, with blueprints giving details of construction. The drawings were made by Mr. F. A. Case of the Bureau, and the instrument constructed in the instrument shop by Mr. H. C. Wunder.

[4] Stokes, C. G., Phil. Mag. (4), *2*, p. 420, (1851).

designed for mounting on the arm of a spectrometer for the analysis of reflected light. The compensator and nicol circles graduated on their cylindrical face (to .25°) are carried respectively on the two ends of a central hub to which is fixed the intervening vernier arms graduated to read hundredths of a degree. Both scales are in the same field of view of the reading telescopes, which are carried by the dust guard for protecting the scales and mounted free of the measuring system. The

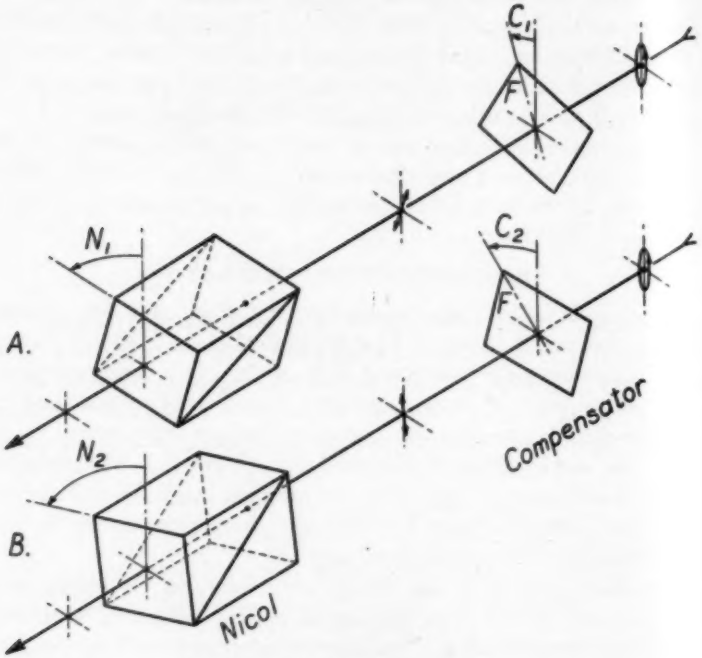


FIG. 5. *The Stokes' elliptic analyzer.*

A. Compensator and nicol are adjusted for extinction and their azimuths recorded.

B. Setting is repeated with compensator at a complementary azimuth.

From their respective changes in azimuth (C and N) the ellipticity is calculated; and from the mean of the two nicol settings the azimuth of the given vibration is obtained.

levers and slow motion screws for rotating the circles are arranged for operation by the right and left hands respectively, as a good setting demands quick alternate adjustment of compensator and analyzing components. Readings may be taken by the operator having the telescopes horizontal as in Fig. 6 or by an assistant with them vertical, as in Fig. 6*a*. The latter plan, by avoiding unnecessary eye fatigue, is

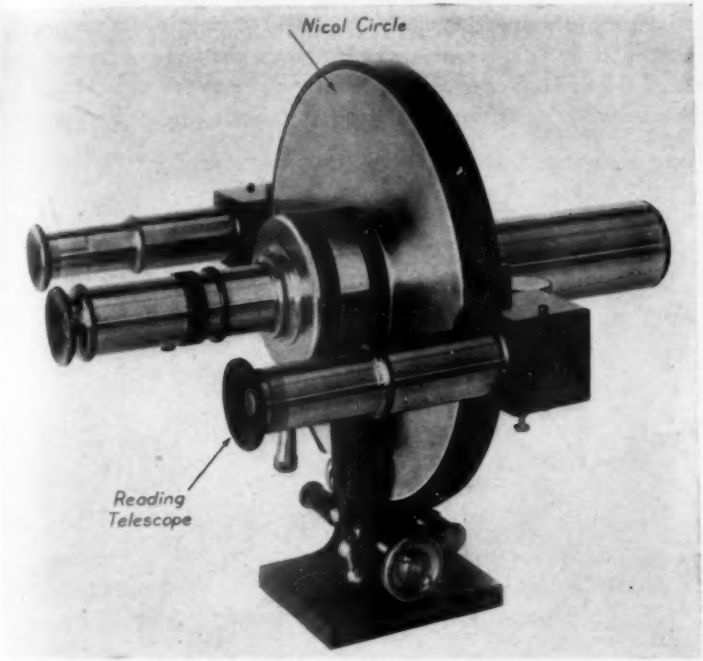


FIG. 6. *Assembled view of universal polarimeter.*

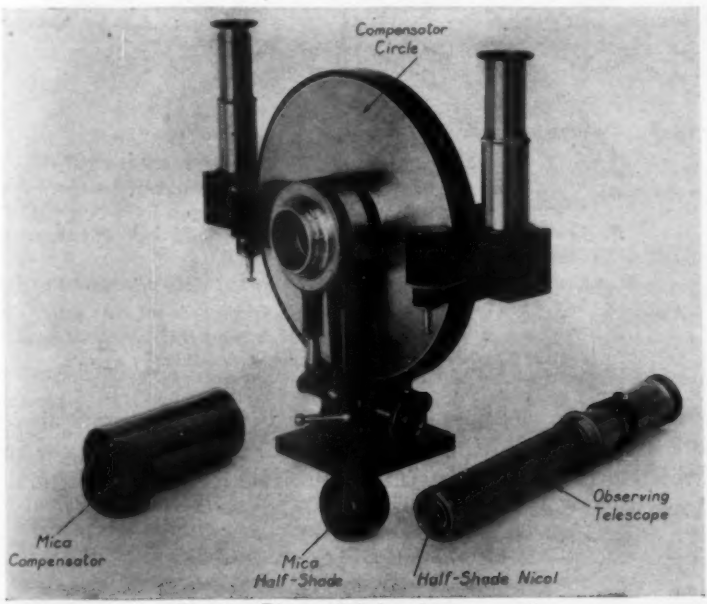


FIG. 6a. *Disassembled view of same reversed.*

very desirable in precision work. The field of view is diaphragmed to a 3 mm aperture at the analyzing nicol, thus giving a convenient size of field in the five-power observing telescope and at the same time largely eliminating errors arising from possible inhomogeneity of polarization

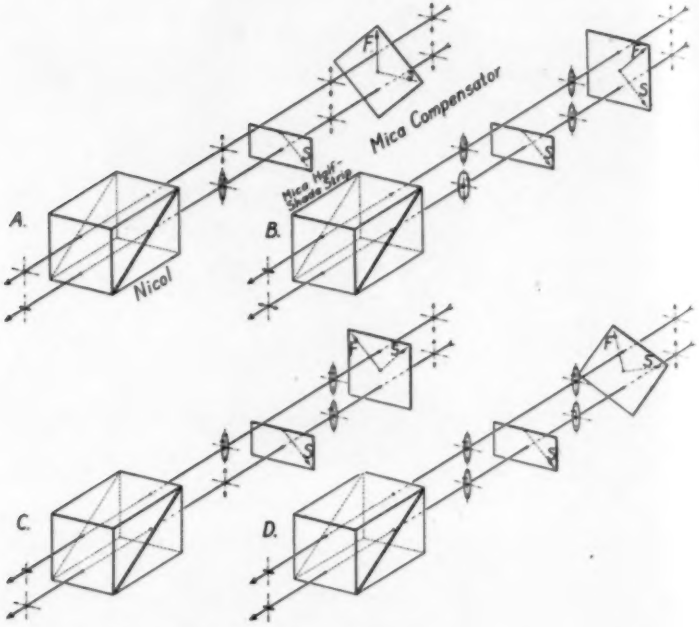


FIG. 7. *The Brace half-shade elliptic analyzer.*

A. Plane polarized entrant light; compensator in neutral position; mica half-shade strip produces positive ellipticity; viewed through nicol lower half of field the brighter.

B. Compensator produces positive ellipticity; mica half-shade strip augments it; lower half of field the brighter.

C. Compensator produces negative ellipticity; half-shade strip restores it; upper half of field the brighter.

D. Half-shade strip converts light received from compensator into an equal and opposite ellipticity of unchanged azimuth—the condition for a match.

across the beam. The use of a square end (Glan-Thompson) nicol in the analyzing system eliminates the orbital displacement of the field on rotating the nicol, which is a disagreeable result when lateral displacement by the nicol is present.

Used as a simple azimuth polarimeter the compensator and elliptic half-shade are omitted, the nicol circle with its half-shade nicol alone being employed.[5]

Employed as a Brace elliptic polarimeter, the optical principles of which are illustrated in Fig. 7, the half-shade nicol is replaced by a

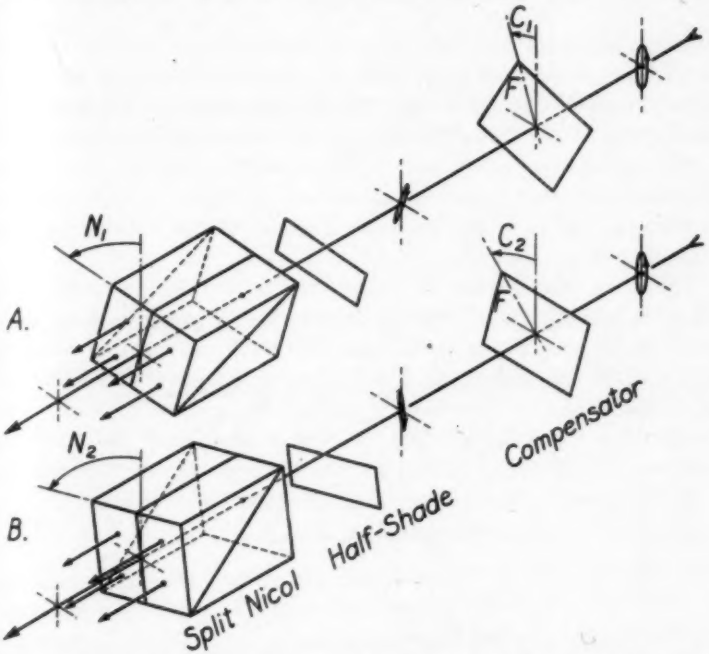


FIG. 8. *Tool's general polarimeter. Replaces simple nicol of Stokes' method with a Cornu-Jellett split nicol and attaches thereto a Brace elliptic half-shade. As in Stokes' method, complementary settings A and B are made in which the four-part field is matched.*

simple nicol, the mica half-shade is mounted in focus and adjusted to the desired azimuth with respect to the nicol, and a thin compensator is used. In this form the nicol circle must be kept clamped in the same position for the entire set of observations, else new zero settings must be made whenever a shift occurs. Equations (23), (24) and (25) serve in this case.

For Tool's method, illustrated by Fig. 8, the Cornu-Jellett half-shade nicol with the Brace elliptic half-shade adjusted to the desired relative

[5] A quartz plate, of known rotatory dispersion, which can be inserted in the compensator tube, is a convenient adjunct for determining the wave length of the light used.

azimuth for a four-part field are brought into simultaneous focus[6] and the quarter-wave or, according to conditions, other compensator used. Equations (26), (27), (28), (30) and (31) are applicable with this form.

II. WORKING THEORY OF THE INSTRUMENT

1. GENERAL

Elliptically polarized light may be distinguished from natural light by viewing it through some kind of a plane polarizer set at various azimuths around the light ray—the different azimuths giving different amplitudes of the transmitted vibration, hence different brightnesses to the field of view. A mixture of plane polarized beams, or, of natural and plane polarized beams will do the same thing but in such a case the brightness does not vary with the azimuth of the viewing polarizer according to the same law.

The direct application of this method to the determination of an elliptical vibration is not feasible, so recourse is had to a bi-refringent plate for "restoring" the given light to a rectilinear, or to some definite nearly rectilinear, vibration. The brightness of the field viewed through an "analyzing nicol" serves as a criterion for a correct setting of the compensator, and the settings of both compensator and analyzing nicol serve to calculate the constants of the given elliptic vibration.

The following relations and designations will be used:

Equation of ellipse referred to its own axes:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

Its "ellipticity," $e=\pm\frac{b}{a}$, is positive, when, looking toward the source, the light vector rotates counter clockwise.

"Order" of a bi-refringent plate, ϕ, is the phase lead in radians which it imparts to the least refrangible (maximum velocity) vibration over the most refrangible (minimum velocity) vibration.

All azimuths, θ, are positive in sign. The azimuth of an ellipse is the smaller angle which its major axis makes with a given direction of reference; that of a rectilinear vibration is the angle which its direction of displacement makes with the given direction; that of a polarizer is the azimuth of its transmitted vibration, while its *extinction azimuth* is

[6] This focus leaves, at best, much to be desired. It needs therefore the most careful adjustment laterally and alignment of the nicol partition, as provided. A minimum length of nicol is also of importance.

at right angles thereto; and that of a bi-refringent plate is that of the least refrangible component.

$$\tan \omega = \frac{b}{a} = e \tag{2}$$

ω being the angle between the major axis of the ellipse and the diagonal of the rectangle having a and b for its sides—(see Fig. 9)

$$\sin 2\omega = \frac{2e}{1+e^2} \tag{3}$$

The locus of a point tracing an elliptical path is given by

$$\begin{aligned} y &= R \sin \frac{2\pi t}{T} \\ x &= A \sin\left(\frac{2\pi t}{T} + \phi\right) \end{aligned} \tag{4}$$

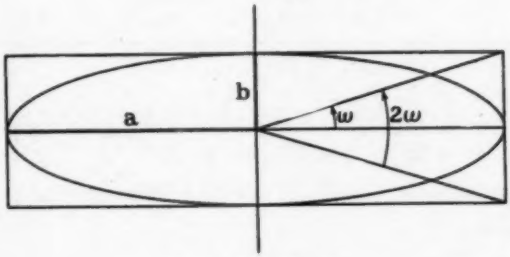


FIG. 9. *Definition of* ω.

in which ϕ gives in radians the phase lead of the x-component of the vibration over the y-component—A and R being the respective amplitudes,[7] T the period of vibration, and t the time elapsed since $y = 0$.

The brightness of a rectilinear vibration is equal to the square of its amplitude, the amplitude being considered as defined by this relation.

Since the differential effect of loss of light by reflection or transmission is negligible in the cases which we treat, we assume in all cases no loss from either.

Our first problem is to express the component amplitude of a given elliptic vibration, such as, for example, the analyzing nicol transmits. From Fig. 10, A is the length of the perpendicular from 0 on the tangent line to the ellipse, $x = A$; and R is likewise that to the tangent line, $y = R$.

We have $$A = x_1' \cos\theta - y_1' \sin\theta$$

[7] R designates the "retarded" and A the "accelerated" component.

where (x_1', y_1') is the point of tangency in the prime system of coordinates and θ the azimuth of the ellipse. Combining this equation with (1) that of the ellipse—

$$\frac{x_1'^2}{a^2}+\frac{y_1'^2}{b^2}=1\,,$$

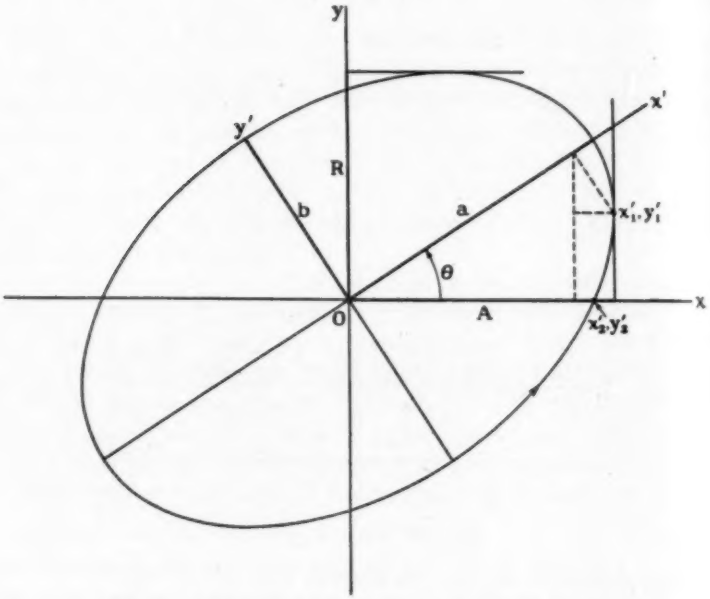


FIG. 10. *Component amplitudes A and R of an elliptic vibration.*

eliminating y_1', solving for x_1', and then imposing the condition that x_1' can have but one root—condition for tangency—we obtain

$$A^2=a^2\cos^2\theta+b^2\sin^2\theta \tag{5}$$

similarly

$$R^2=a^2\sin^2\theta+b^2\cos^2\theta \tag{6}$$

By adding and subtracting (5) and (6) two important relations are obtained which will be used later, namely

$$A^2+R^2=a^2+b^2 \tag{7}$$

$$A^2-R^2=(a^2-b^2)\,(\cos^2\theta-\sin^2\theta)=(a^2-b^2)\cos 2\theta \tag{8}$$

Equation (7) states the obvious fact that the sum of the two component brightnesses is independent of the azimuth of the ellipse, being equal to the sum of the major and minor brightness components; while (8) states

that the difference in the component brightnesses varies as the cosine of twice the azimuth of the ellipse.

By introducing the phase lead ϕ, of the A-component over the R-component, the two other, of four, fundamental relations needed here are derived. Introducing in (4) the condition $y=0$ and $x=p$ we have for the intercept on the x-axis

$$p=A\sin\phi \tag{9}$$

The coordinates of this point in the prime system being designated by x_2' and y_2' we have from Fig. 10

$$x_2'=p\cos\theta=A\sin\phi\cos\theta$$

$$y_2'=p\sin\theta=A\sin\phi\sin\theta$$

which substituted in (1) gives

$$A^2\sin^2\phi=\frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta} \tag{10}$$

and from (6)

$$A\,R\sin\phi=ab \tag{11}$$

We have further

$$\cos^2\phi=1-\sin^2\phi=1-\frac{a^2b^2}{(a^2\cos^2\theta+b^2\sin^2\theta)\,(a^2\sin^2\theta+b^2\cos^2\theta)}$$

by substitution from (10) and (5). This may be written

$$\cos^2\phi=1-\frac{a^2b^2}{[a^2-(a^2-b^2)\sin^2\theta]\,[a^2-(a^2-b^2)\cos^2\theta]}=\frac{(a^2-b^2)^2\sin^2\theta\cos^2\theta}{A^2R^2}$$

hence

$$AR\cos\phi=\frac{a^2-b^2}{2}\sin 2\theta\,. \tag{12}$$

Equations (7), (8), (11), and (12), the four fundamental relations referred to above, will for convenience be rewritten as a group, following Tuckerman's notation.[8]

$$\left.\begin{aligned}
P&\equiv\frac{A^2+R^2}{2}=\frac{a^2+b^2}{2} && (a)\\
Q&\equiv\frac{A^2-R^2}{2}=\frac{a^2-b^2}{2}\cos 2\theta && (b)\\
K&\equiv AR\cos\phi=\frac{a^2-b^2}{2}\sin 2\theta && (c)\\
S&\equiv AR\sin\phi=ab && (d)
\end{aligned}\right\} I$$

[8] Tuckerman, L. B., "Doubly Refracting Plates and Elliptic Analyzers," Univ. of Neb. Studies IX; April 1909.

Tuckerman has shown that by expressing the vibration in terms of P, Q, K, and S the effect of a bi-refringent plate, or series of plates, may be readily written. The effect of such a plate is easily visualized by recognizing that, as he shows, Q, K, and S may be considered as the rectangular coordinates of a point on the surface of a sphere of radius P, the effect of the plate being merely to shift this point on the surface by rotation about a definite axis. For we find from (I) that

$$Q^2+K^2+S^2=P^2$$

To show how these may be used to determine the constants of an elliptic vibration, we have for the azimuth θ of the ellipse, obtained from I (b) and I (c),

$$tg\ 2\theta=\frac{K}{Q}$$

in which θ can obviously assume any value between 0 and π and is independent of S. Likewise, independent of Q and K, the ellipticity is derived for example from I (a) and I (d)

$$\frac{S}{P}=\frac{2ab}{a^2+b^2}$$

$$=\frac{2e}{1+e^2}$$

and from (3)

$$=\sin 2\omega \tag{13}$$

and finally $$e=\tan\omega$$

according to (2) in which, since $\sin 2\omega$ can obviously assume any value between -1 and $+1$, e may range anywhere between -1 and $+1$. Further the phase lead ϕ of any A-component, from I (c) and I (d) is given by

$$\tan\phi=\frac{S}{K}$$

Let, in Fig. 11, the Q-K plane be the equatorial plane of the sphere and the S-axis be the polar axis. Let the longitude be measured from the Q-axis eastward, and the latitude upward from the equatorial plane. Inspection shows that 2θ is the longitude and 2ω is the latitude of the expressed vibration. Thus its longitude on this sphere of reference can be used to determine the azimuth of the vibration while its latitude can be used to determine the ellipticity. Plane polarized light is represented by the sum total of points on the equator while positive

and negative circularly polarized light are represented respectively by the north and south poles.

2. ANALYTICAL TREATMENT FOR A VARIABLE AZIMUTH COMPENSATOR

To calculate the effect of a given bi-refringent plate set at some definite azimuth, we take the given light as referred to the basic system of coordinates in Fig. 10, transform it to a new system of coordinates having the x-axis parallel to the azimuth of the plate, then add to the existing phase lead (or lag) of the x-component vibration the additional phase lead impressed by the plate. The resulting vibration is then transformed back to the original system of coordinates.

Take a vibration such as expressed in equations I.

$$\left.\begin{aligned} P_0 &= \frac{a_0^2+b_0^2}{2} && (a)\\ Q_0 &= \frac{a_0^2-b_0^2}{2}\cos 2\theta_0 && (b)\\ K_0 &= \frac{a_0^2-b_0^2}{2}\sin 2\theta_0 && (c)\\ S_0 &= a_0 b_0 && (d) \end{aligned}\right\} I_0$$

This light is now supposed to be modified by passing through a bi-refringent plate set at an azimuth θ with respect to the basic system. It is therefore referred to this new system of coordinates, the new xy-system making an angle θ with the basic. Similarly with the sphere, the new QK system of reference makes an angle 2θ with the basic, i.e., $\sphericalangle Q,Q' = 2\theta, \sphericalangle K,K' = 2\theta, \sphericalangle S,S' = 0$. We have the new coordinates Q_0', K_0', S_0', of the point, expressed in terms of the old

$$\left.\begin{aligned} P_0' &= P_0 = \frac{a_0^2+b_0^2}{2} && (a)\\ Q_0' &= Q_0\cos 2\theta + K_0 \sin 2\theta = \frac{a_0^2-b_0^2}{2}(\cos 2\theta_0 \cos 2\theta + \sin 2\theta_0 \sin 2\theta)\\ &= \frac{a_0^2-b_0^2}{2}\cos 2(\theta_0-\theta) && (b)\\ K_0' &= -Q_0 \sin 2\theta + K_0 \cos 2\theta = \frac{a_0^2-b_0^2}{2}\sin 2(\theta_0-\theta) && (c)\\ S_0' &= S_0 = a_0 b_0 && (d) \end{aligned}\right\} I_0'$$

This transformation is checked by forming the sum of the squares of the new coordinates of the point, namely

$$Q_0'^2+K_0'^2+S_0'^2=Q_0^2+K_0^2+S_0^2=\frac{a_0^2+b_0^2}{2}$$

To obtain the effect of transmission through the bi-refringent plate we need now, as already stated, simply to add the phase lead ϕ which it impresses to the existing phase lead as expressed in the new system. This latter is not explicitly given but might be derived by using the AR terms of I. The derivation however, is unnecessary if we note that the addition of ϕ to ϕ_0 rotates the given point through the angle ϕ about the Q axis—leaving the Q coordinate unchanged. To show this, we have by adding ϕ to the initial phase ϕ_0 of I, for example

$$\begin{aligned} Q' &= Q \\ K' &= AR\cos(\phi_0+\phi) \\ &= AR(\cos\phi_0\cos\phi-\sin\phi_0\sin\phi) \\ &= K\cos\phi-S\sin\phi \end{aligned}$$

Likewise $S'=K\sin\phi+S\cos\phi$

in which K' and S' together show a rotation of the point (K,S) through the angle ϕ in the KS plane—that is, about Q as an axis. Returning to our specific problem, the new position $(Q_1', K_1', S_1',)$ on the sphere, is then

$$\left.\begin{aligned} P_1' &= P_0' = \frac{a_0^2+b_0^2}{2} && (a) \\ Q_1' &= Q_0' = \frac{a_0^2-b_0^2}{2}\cos 2(\theta_0-\theta) && (b) \\ K_1' &= K_0'\cos\phi-S_0'\sin\phi = \frac{a_0^2-b_0^2}{2}\sin 2(\theta_0-\theta)\cos\phi-a_0b_0\sin\phi && (c) \\ S_1' &= K_0'\sin\phi+S_0'\cos\phi = \frac{a_0^2-b_0^2}{2}\sin 2(\theta_0-\theta)\sin\phi+a_0b_0\cos\phi && (d) \end{aligned}\right\} I_1'$$

Checking this set as before we find

$$Q_1'^2+K_1'^2+S_1'^2=Q_0'^2+K_0'^2+S_0'^2=P_0'^2=P_0^2$$

which shows that the new position of the point is on the surface of the sphere, hence all positions so obtained.

To return to the basic system of reference we now rotate the $Q'K'$ axes about the S' axis back through the angle -2θ and have the new

position of the point expressed in the basic system of coordinates. We can, however, forego carrying through this transformation in detail since from (13) we need, for specifying the ellipticity of the resultant vibration, only the S_1 and P_1 values in the basic system. The transformation being a rotation about the S_1'-axis, we have obviously from I_1' (d)

$$S_1 = S_1' = \frac{a_0^2 - b_0^2}{2} \sin 2(\theta_0 - \theta) \sin\phi + a_0 b_0 \cos\phi \qquad I_1\ (d)$$

and, in general,

$$P_1 = P_1' = - \quad - \quad - \quad = P_0 \qquad I_1\ (a)$$

These equations serve for deriving the formulas (23), (24), and (25) used in connection with the Brace method.

3. GEOMETRICAL TREATMENT FOR A VARIABLE AZIMUTH COMPENSATOR

In Fig. 11, the sphere is used to represent the effect of a bi-refringent plate set at an azimuth θ—longitude $QOQ' = 2\theta$—and impressing a phase lead ϕ. Points on the great circle $M\ M'\ M''\ M'''$ are thereby restored to corresponding positions on the equator, that is, to rectilinear vibrations having the corresponding azimuths. Other points such as u are rotated about the same axis Q' and through the same angle to v. Points in general differing in longitude by 180°, in azimuth by 90°, are equally but oppositely affected. Points having a longitude less than the axis of rotation Q' are decreased in ellipticity while those having a longitude greater than Q' are increased. This fact is important in determining the direction of vibration. Points having a longitude differing by 90° from that of the bi-refringent plate, azimuth at 45° to it, suffer a maximum change in latitude (or ellipticity) and a zero change in longitude (or azimuth). The two points on the Q'-axis—plane polarized light having same azimuth as the plate or 90° thereto—alone suffer no change.

Fig. 12 represents the geometrical relations involved in restoring elliptically polarized light at M (latitude $2\omega_0$, longitude H) to a rectilinear vibration at longitude H_1 by means of a bi-refringent plate of order $\phi = \sphericalangle MTH_1$ set at the necessary longitude Q'. The circular arc MH_1, generated by rotation about the Q' axis, indicates the change in the ellipticity and azimuth of the vibration as the light proceeds through the compensating plate. B being the projection of M on the equatorial plane, we have from the indicated construction of the figure

$$\cos 2\omega_0 \equiv \cos\sphericalangle HOM = \frac{OB}{OM}$$

$$= \frac{OB}{OH_1}$$

$$= \frac{\sin\sphericalangle OH_1B}{\sin\sphericalangle OBH_1} \qquad \text{in } \Delta OH_1B$$

$$= \frac{\sin(\sphericalangle H_1BH - \sphericalangle H_1OB)}{\sin\sphericalangle H_1BH}$$

since $\sphericalangle H_1BH + \sphericalangle H_1BO = \sphericalangle H_1BO + \sphericalangle H_1OB + \sphericalangle OH_1B$

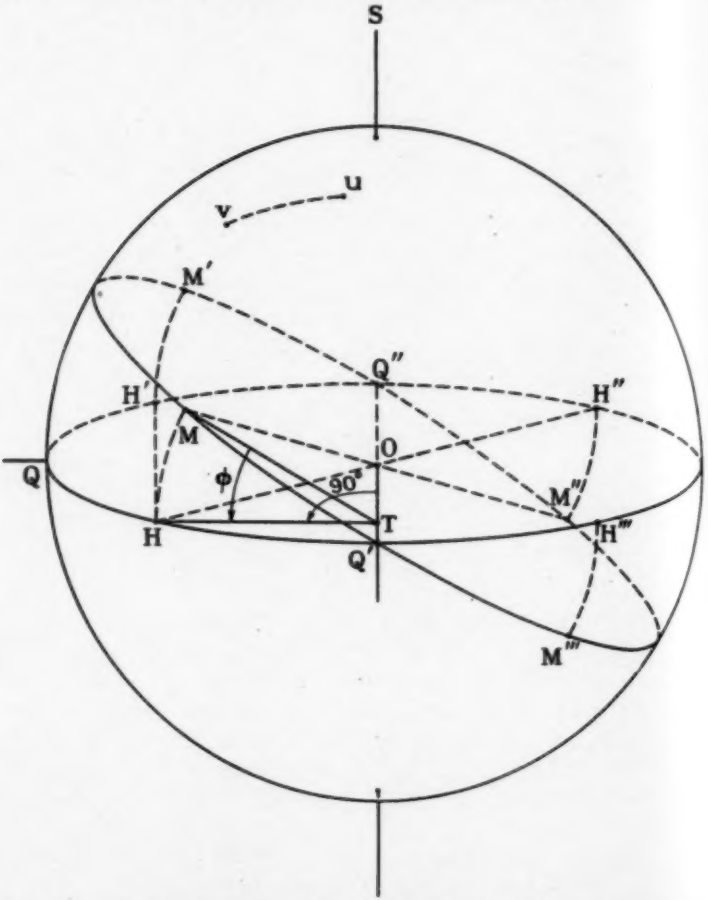


FIG. 11. *An elliptic vibration represented by locus of a point on the surface of a sphere; shows the effect of a birefringent plate of order ϕ. This shifts M to H; M′, having same ellipticity, to H′; and in general u to v.*

Substituting for short, the notation indicated, this gives

$$\cos 2\omega_0 = \frac{\sin(c_0 - n_0)}{\sin c_0} \tag{14}$$

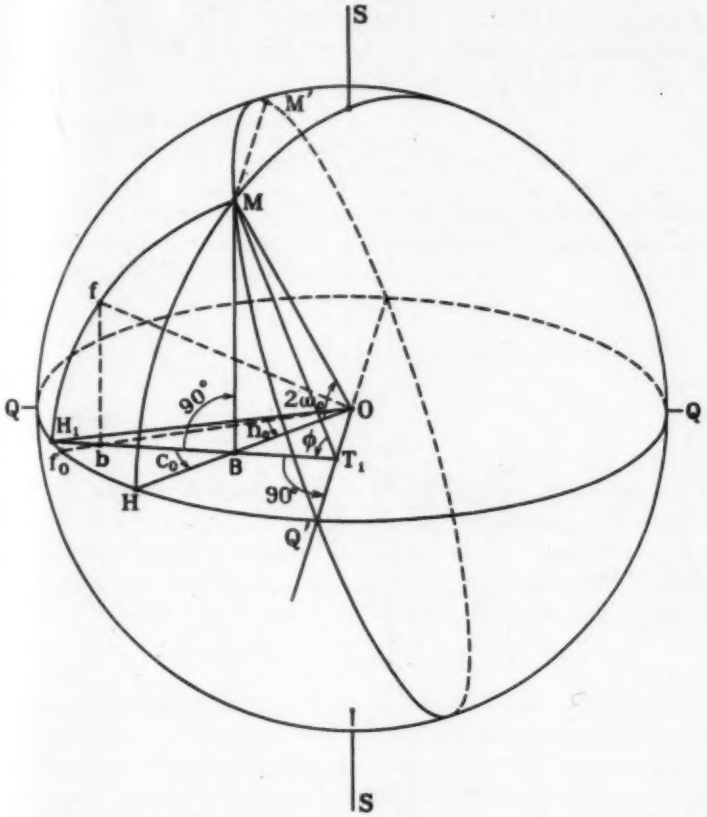


FIG. 12. *Geometrical derivation of Stokes' formulas. The given vibration M (at longitude H) is restored by the compensator, of order φ, set at an azimuth θ₀' (longitude, 2θ₀'), to a rectilinear vibration at H₁.*

We have further

$$\cos\phi \equiv \cos\sphericalangle H_1T_1M = \frac{BT_1}{MT_1}$$

$$= \frac{BT_1}{H_1T_1}$$

$$= \frac{OT_1/H_1T_1}{OT_1/BT_1}$$

$$= \frac{\tan\sphericalangle OH_1B}{\tan\sphericalangle H_1BH}$$

hence as above,

$$\cos\phi = \frac{\tan(c_0 - n_0)}{\tan c_0} \tag{15}$$

By measuring the complementary azimuths of the compensator—the second azimuth, for instance, being such that the point M' of the compensator trace is brought to the position M on the sphere—and the

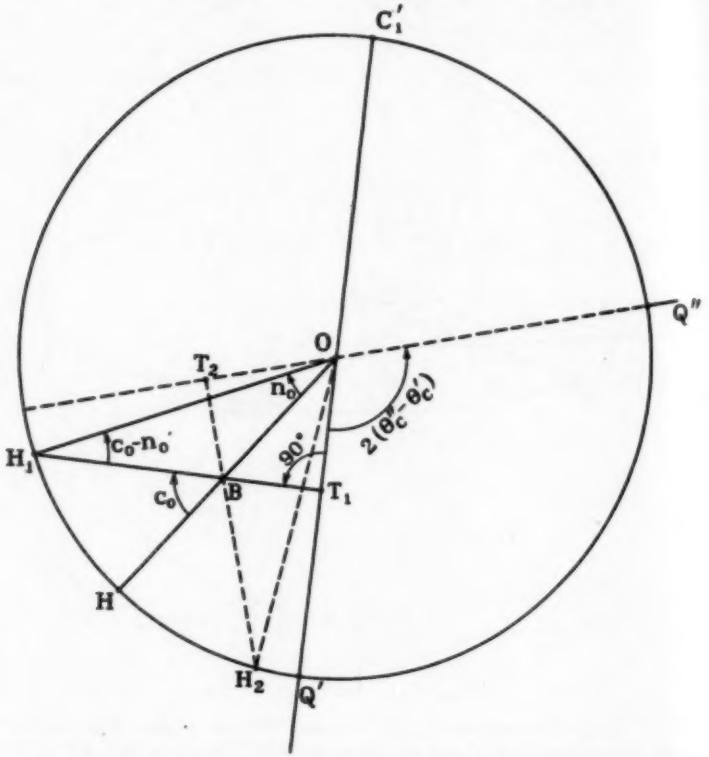


FIG. 13. *Trace of Fig. 12 on equatorial plane showing also the complementary settings, Q'' and H_2 respectively of compensator and analyzing nicol, obtained by shifting the compensator so that M' on its circle Q'MM' falls at M on the sphere and consequently rotates the given vibration through the same angle ϕ to the position H_2 on the equator.*

corresponding azimuths of the analyzing nicol set for extinction on the restored light, Stokes calculated from expressions similar to (14) and (15) the ellipticity and azimuth of the initial vibration and also the order of the compensator. The trace on the equatorial plane of these two

complementary settings, one in full line, the other dashed, is given in Fig. 13. The full lines are taken directly from Fig. 12, the dashed are corresponding lines obtained by considering the compensator to be rotated through a longitude $\sphericalangle Q'OQ'' = 2(\theta_c'' - \theta_c')$ so that M' of Fig. 12 coincides with M, and the light is restored to H_2 of Fig. 13 instead of H_1. The extinction azimuths of the analyzing nicol correspond to the longitudes H_1 and H_2 on the sphere such that $\sphericalangle H_1OH_2 = 2(\theta_n - \theta'_n)$. From the construction indicated

$$\sphericalangle H_1BH_2 = 2(\theta_c'' - \theta_c')$$

Furthermore since from symmetry $H_2T_2 = H_1T_1$, and $OT_2 = OT_1$, the line OH, passing through the intersection at B, bisects the angles H_1OH_2 and H_1BH_2. Therefore

$$\begin{aligned} c_0 &\equiv \sphericalangle H_1BH \\ &= \tfrac{1}{2}\sphericalangle H_1BH_2 \\ &= \theta_c'' - \theta_c' \end{aligned} \tag{16}$$

and similarly

$$n_0 = = \theta_n'' - \theta_n' \tag{17}$$

Stokes method does not, however, give the desired accuracy of setting either in restoring the light or in determining the azimuth of the restored vibration. The modification of his method by Tool which, making use of the half-shade principle and almost the same formulas as given in (14) and (15), gives high accuracy will be treated under Applications.

4. APPLICATIONS TO ELLIPTIC ANALYZING SYSTEMS

These systems depend in general on the equality of brightness of two different parts of the field of view as observed through a polarizer. The problem therefore is to express the conditions under which vibrations differing in general in latitude and longitude will give the same brightness to the component transmitted by the observing nicol. Let A_n^2 and R_n^2 be respectively the component brightnesses of the light incident on the analyzing nicol, the one parallel to the transmitted vibration, the other parallel to the extinguished.

From (I)

$$P_n = \frac{A_n^2 + R_n^2}{2} = \text{const}$$

$$Q_n = \frac{A_n^2 - R_n^2}{2}$$

The imposed condition is that $A_n^2 = \text{const}$, hence also $R_n^2 = \text{const}$. It follows therefore $Q_n = \text{const}$ (18)

The interpretation is that every plane normal to the Q_n-axis cuts the spherical surface in a circle whose points represent equal brightnesses when viewed through a polarizer whose azimuth (or extinction azimuth) corresponds to the Q_n-axis. In other words: given the longitude of the analyzing nicol—for either the transmitted or the extinguished component—all circles on the sphere having this longitude as an axis represent vibrations giving equal brightness to the field as seen through the nicol.

Dealing with the extinction azimuth of the nicol, the observed brightness

$$R_n^2 = P_n - Q_n \tag{19}$$

is a minimum equal to zero for $Q_n = P_n$; that is, for a rectilinear vibration at the extinction azimuth of the nicol, the brightness of field is zero. It is a maximum, equal to $2P_n$, for $Q_n = -P_n$, that is, for a rectilinear vibration at a longitude 180° from the nicol longitude, 90° from the extinction azimuth of the nicol. Both are obvious conclusions.

Method of Isochromatic Lines: This is taken up here to serve as a very simple example of the applicability of this method of analysis. The device was first used by Maxwell[9] in making photoelastic analyses. In this a quarter-wave plate, which impresses a phase lead of $\frac{\pi}{2}$ is fixed to the polarizer at an azimuth of 45° thereto—giving circularly polarized light, represented say by the north pole $Q = K = 0$, $S = P$. Another quarter-wave plate is similarly fixed to the analyzing nicol at 45°. Between the two quarter-wave plates is inserted the stressed plate. Those parts of the stressed plate impressing the same phase lead, irrespective of azimuth considerations, appear equally bright when viewed through the analyzing nicol. Wherever the strained plate impresses a phase lead, say ϕ, on the given circularly polarized light, it rotates the polar point about some Q-axis lying in the equatorial plane, through an angle ϕ, to some point on the circle MM' at latitude $(\frac{\pi}{2} - \phi)$, Fig. 14. The quarter-wave plate fixed to the analyzing nicol and having a longitude Q_e rotates this circle, about the axis Q_e, through an angle of 90° so that the axis of the circle is brought into the equatorial plane at 90° longitude to that, Q_e, of the quarter-wave plate, and therefore coincident in longitude with the Q_n-axis of the analyzing nicol. The circle in its final position is expressed therefore by

$$Q_n = P_n \cos \phi$$

[9] Coll. Sci. Papers *I* p 68.

hence is an equal brightness circle as viewed through the nicol. The brightness is, from (19)

$$R_n^2 = P_n - Q_n$$
$$= P_n(1 - \cos\phi) \qquad (20)$$

When white light is used, the different wave-lengths have different values for ϕ, hence the position on the stressed plate of maximum bright-

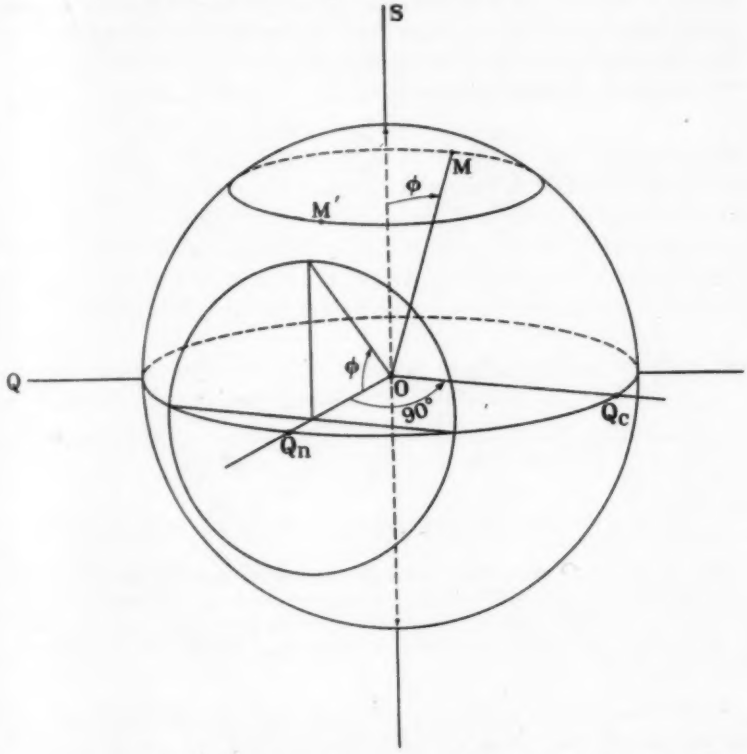


FIG. 14. *Analysis of method of isochromatic lines used in photoelastic analyses. All parts of the stressed plate which impress the same phase lead ϕ, irrespective of azimuth, convert the entrant circular polarized vibration, located on north pole S, to points on the latitude circle MM'. A second quarter-wave plate at longitude Q_c rotates this circle, about OQ_c through a right angle, so that its axis falls in the equatorial plane and coincident in longitude Q_n with the analyzing nicol. This is the criterion for equal brightness of field as seen through the nicol.*

ness of one color differs in general from that of the others, the result being that the plate exhibits a pattern of isochromatic lines. It is

obvious that a rotation of the stressed plate in its own plane should carry the pattern of colors with it, and that a rotation of the analyzing system leaves it unchanged. This is not strictly true in practice owing to the lack of achromatic quarter-wave plates.

Elliptic Half-Shades: These all function by furnishing to the analyzing nicol contiguous parts of the field of view in which the light to be measured is restored to different but definite, preferably small, ellipticities. Restored light having a large ellipticity gives, with a satisfactory source, a high brightness level to the field and consequently a low precision in setting. The criterion for a correct setting is a brightness match of these contiguous parts.

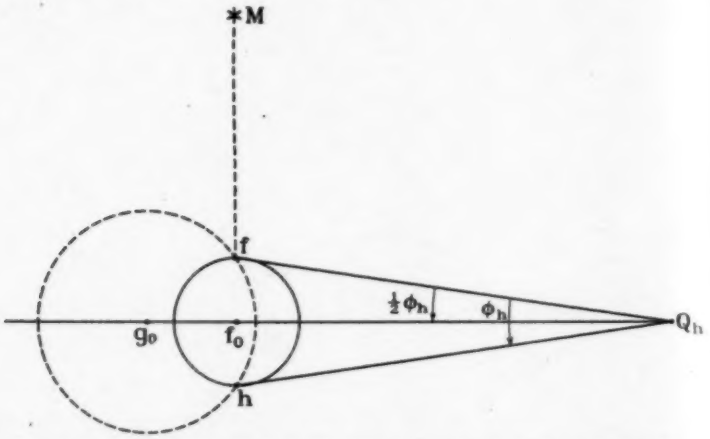


FIG. 15. *Analysis of Brace method. The figure represents a projection in part of Fig. 12 on a plane perpendicular to* Of_0. *The line* $Q_h f_0 g_0$ *represents the equator. M, f and h represent points on the surface of the sphere; the angle marked* ϕ_h *is therefore the projected order of the half-shade, not the true order.* Q_h *represents the longitude of the half-shade, similar to that,* Q_c, *of the compensator in Fig. 12.*

Tuckerman[10] has given a very complete discussion of the theory of the various methods. The aim in the presentation here will be to make that theory simpler, and the operations more easily visualized but no less exact, though at the expense of both elegance of presentation and generality of procedure.

Fig. 15 shows the sphere analysis applied to the Brace elliptic half-shade. The given light M is restored by the compensator to a position f such that the half-shade at azimuth Q_h, and order ϕ_h, carries it, on that

[10] Loc. cit.

part of the field, to another equal brightness point h. Thus f and h lie respectively at the two intersections of a circle $Q_n = \text{const}$ with another having Q_h (lying also in equatorial plane) as axis. Symmetry requires that f and h have the same longitude, equal and opposite latitudes. Any circle on the sphere whose axis lies in the equatorial plane can be shifted to an axial position g_0 such that this condition obtains—the field is therefore matched for all positions of the analyzing nicol. However, the most sensitive azimuth for the analyzing nicol is f_0, having same longitude as f and h, because the brightness change arising from an incorrect latitude setting of the fh pair is a maximum when the shift takes place in the direction of the radius, and that radius is the minimum consistent with a satisfactory brightness level.

With the half-shade plate fixed at Q_h relative to the Nicol at f_0, the latitude of f, for a match is constant, irrespective of the latitude of M. The relative longitude of M and ff_0h can always be adjusted, without disturbing the measuring system so that the vibration is restored to f—either by varying the azimuth of M to give minimum brightness of field, or by shifting that of the analyzing system as a unit so as to do the same thing.

Equations I_1 (d) and (a) are readily applied here to calculate the ellipticity e_0 of the given vibration M. We have

$$(S_1)_f = \frac{a_0^2 - b_0^2}{2} \sin 2(\theta_0 - \theta) \sin\phi + a_0 b_0 \cos\phi$$

Letting $2\omega_f$ be the latitude of f, and recalling that $P_1 = P_0 = \frac{a_0^2 + b_0^2}{2}$ is the radius of the sphere, we have from (13)

$$\sin 2\omega_f = \frac{(S_1)_f}{P_0} = \frac{a_0^2 - b_0^2}{a_0^2 + b_0^2} \sin 2(\theta_0 - \theta) \sin\phi + \frac{2a_0 b_0}{a_0^2 + b_0^2} \cos\phi$$

or

$$\frac{\sin 2\omega_f}{\cos\phi} = \frac{1 - e_0^2}{1 + e_0^2} \sin 2(\theta_0 - \theta) \tan\phi + \frac{2e_0}{1 + e_0^2} \qquad (21)$$

The left hand member of this equation being a constant of the measuring system, the relation holds when M is a rectilinear vibration, that is for $e_0 = 0$. Substituting $e_0 = 0$ in (21) and letting θ' be the observed value of θ for this setting, we obtain the value of the constant

$$\frac{\sin 2\omega_f}{\cos\phi} = \sin 2(\theta_0 - \theta') \tan\phi \qquad (22)$$

Introducing this in (21), substituting for covenience

$k_1 = \tan\phi\,[\sin 2(\theta_0-\theta) - \sin 2(\theta_0-\theta')]$

$k_2 = \tan\phi\,[\sin 2(\theta_0-\theta) + \sin 2(\theta_0-\theta')]$, and solving for e_0

$$e_0 = \frac{1}{k_2}(1 \pm \sqrt{1+k_1k_2})$$

Recalling that $e_0 \gtrless 1$ the negative sign of the radical must be chosen. Furthermore since $k_1k_2 < 1$, we may expand the radical term by the binomial theorem and obtain a convergent series. This reduces to

$$e_0 = -\frac{k_1}{2}\left(1 - \frac{k_1k_2}{4} + \frac{k_1^2k_2^2}{8} - \quad - \quad - \quad -\right) \tag{23}$$

We have had in actual practice no occasion to use any but the first term of this series, the second term never having been more than one-tenth of a percent correction on the first.

The expression

$$\begin{aligned} e_0 &= -\frac{k_1}{2} \\ &= -\tfrac{1}{2}\tan\phi\,[\sin 2(\theta_0-\theta) - \sin 2(\theta_0-\theta')] \\ &= \tfrac{1}{2}\tan\phi\,[\sin 2(\theta-\theta_0) + \sin 2(\theta_0-\theta')] \end{aligned} \tag{24}$$

is sufficiently exact for most practical cases. In (24) there remains however θ_0 to be determined. If in (22), obtained with plane polarized light, a complementary setting of the compensator for a match be taken, its azimuth being θ'', we would have

$$\sin 2(\theta_0-\theta'') = \sin 2(\theta_0-\theta')$$

or

$$2(\theta_0-\theta'') = -\pi + 2(\theta_0-\theta')\,,$$

(since $\theta'' > \theta_0$ by the Fig. and $\theta_0 > \theta'$ by arbitrary choice) from which

$$\theta_0 = \frac{\theta''-\theta'}{2} - \frac{\pi}{4} \tag{25}$$

It should be noted in connection with this arbitrary choice that a given compensator has a wider range if the half-shade acts with it, as indicated in Fig. 15, instead of acting in opposition thereto. If the azimuths of half-shade and compensator be so chosen for example that the compensator brings M to h and the half-shade plate on its part of the field carries it back to f, the maximum ellipticity which the compensator will match is less than the arrangement shown will do.

It should also be noted that by changing the longitude of the half-shade with respect to that of the analyzing nicol, the brightness level of the matched field can be varied from zero to a maximum without impairing in the least, the applicability of (23) and (24), though to each azimuth of the half-shade there must of course, be a special value of θ'.

Tool's Modification: The optical arrangement is the same as for the Brace method with a Cornu-Jellett half-shade nicol instead of the simple analyzing nicol. Fig. 16 gives the sphere diagram showing but a slight modification from that used for analyzing the Brace instrument. A match of the four parts of the field results when the restored vibrations f and h fall respectively on the two intersections of the equal brightness circles of the two parts of the nicol—axes at f_1 and f_2. The longitudes of f_1 and f_2 having a fixed separation, any shift of their

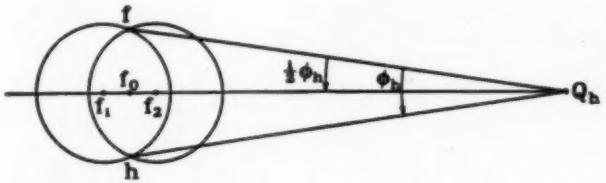


FIG. 16. *Analysis of Tool's method—similar to Fig. 15.*

middle point from f_0 changes the brightness of the two nicol fields oppositely by simultaneously increasing the diameter of one of the circles through f and decreasing the other. Similarly any shift of the fh pair in latitude changes oppositely the brightness of the two parts composing the elliptic half-shade field. Thus we have here a sensitive criterion for correct adjustment of both the compensator and the nicol.

From Figs. 12 and 13 equation (14) was derived and settings assumed to be made on the condition that the given light M was restored by the by the compensator to a rectilinear vibration at H_1 (or H_2). If now it be only partially restored, to f instead of H_1, the nicol setting falls at f_0 instead of H_1. That is, for the same value of $c = c_0$ the nicol value n is smaller than n_0 of (14). Dropping a perpendicular from f on the equatorial plane it falls at b on H_1T_1 (Figs. 12 and 17). Let $\sphericalangle HOb = n$ and $\sphericalangle HOH_1 = n_0$.

Since

$$\sphericalangle OH_1T_1 = c_0 - n_0$$

$$\sphericalangle ObT_1 = c - n$$

In ΔOH_1b

$$\frac{\sin(c_0-n_0)}{\sin \measuredangle H_1bO}=\frac{Ob}{OH_1}$$

$$=\frac{Ob}{Of} \qquad \text{(of Fig. 12)}$$

or

$$\frac{\sin(c_0-n_0)}{\sin(c-n)}=\cos 2\omega_f$$

$2\omega_f$ being the latitude of f. Substituting in (14) for $\sin(c_0-n_0)$ and also $\sin c$ for $\sin c_0$

$$\cos 2\omega_0 = =\frac{\sin(c-n)}{\sin c}\cos 2\omega_f \tag{26}$$

and likewise for the order ϕ of the compensator,

$$\cos \phi=\frac{\tan(c_0-n_0)}{\tan c_0}$$

$$=\frac{\tan(c-n)}{\tan c\sqrt{1+\dfrac{\tan^2 2\omega_f}{\cos^2(c-n)}}} \tag{27}$$

The azimuth of M is obviously obtained as the mean of the two complementary nicol settings.

To obtain $\cos 2\omega_f$ of (26) complementary settings c_1 and n_1 are made on plane polarized light for which $\cos 2\omega_0 = 1$
hence

$$\cos 2\omega_f=\frac{\sin c_1}{\sin(c_1-n_1)} \tag{28}$$

n_1 here being negative in sign to c_1. With the half-shade plate at an azimuth 45° to the analyzing nicol, at a longitude therefore of 90° thereto, T_1 of Fig. 17 falls at 0, and b falls on the radius OH_1. For such an adjustment $\frac{1}{2}\phi_h$ becomes equal to the latitude, $2\omega_f$, of f, hence

$$\cos \tfrac{1}{2}\phi_h=(\cos 2\omega_f)_{max} \tag{29}$$

This relation furnishes a method for evaluating the order ϕ_h of the half-shade plate.

The azimuth θ_0 of the given vibration is obviously at 90° to the mean θ_n of the complementary nicol settings θ_n' and θ_n'' (extinction azimuth) when matching, that is

$$\theta_0=\frac{\theta_n'+\theta_n''}{2}\pm\frac{\pi}{2} \tag{30}$$

Finally, as pointed out under the geometrical treatment, the sign of the measured ellipticity is *positive* when, using the mean, θ_c, of the complementary settings of the compensator,

$$0<(\theta_c-\theta_0)<\frac{\pi}{2}$$

and *negative* when

$$0>(\theta_c-\theta_0)>-\frac{\pi}{2} \qquad (31)$$

For applying this test a shift of π in azimuth is permissible with either θ_c or θ_0 if their calculated difference is greater in magnitude than $\frac{\pi}{2}$.

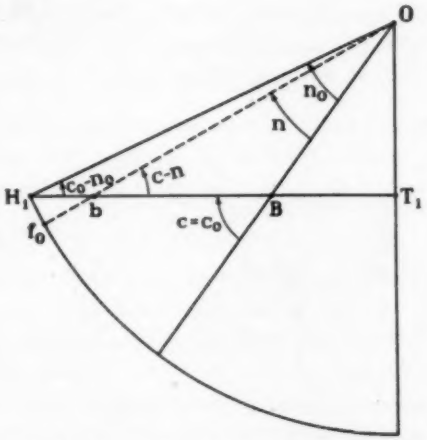


FIG. 17. *Geometrical diagram for deriving formulas used in Tool's method.*

To summarize, the instrument described may be readily used:

(a) As an azimuth polarimeter, for measuring rotation of plane polarized light;

(b) As a Brace elliptic polarimeter, for measuring small ellipticities —but not azimuths—in which the ellipticity sought, e_0, is given with sufficient accuracy by the equation

$$e_0=\tfrac{1}{2}\tan\phi[\sin 2(\theta-\theta_0)+\sin 2(\theta_0-\theta')] \qquad (24)$$

where ϕ is the order of the compensator, θ the azimuth of the compensator for a match on the given light (e_0), θ' the same for a match on plane polarized light, and

$$\theta_0=\frac{\theta''-\theta'}{2}-\frac{\pi}{4}$$

θ'' being the complementary setting on the plane polarized light at an azimuth giving minimum field brightness. Equation (24) assumes the half-shade to act with, not opposite to the compensator. For higher precision in calculation of e_0 (23) may be used instead of (24).

(c) As a Tool general polarimeter, for measuring both azimuth and ellipticity, in which the ellipticity sought (e_0) is derived from the equations

$$\cos 2\omega_0 = \frac{\sin(c-n)}{\sin c}\cos 2\omega_f \tag{26}$$

$$e_0 = \tan\omega_0$$

where c and n denote the change respectively in compensator and nicol azimuths between complementary settings for a match of the four-part field. The constant $\cos 2\omega_f$ is obtained from corresponding complementary settings c_1 and n_1 on plane polarized light of any azimuth, using the equation

$$\cos 2\omega_f = \frac{\sin c_1}{\sin(c_1-n_1)} \tag{28}$$

in which n_1 must obviously be negative to c_1.

The azimuth, θ_0, of the given vibration is calculated from

$$\theta_0 = \frac{\theta_n' + \theta_n''}{2} \pm \frac{\pi}{2} \tag{30}$$

where θ_n' and θ_n'' are the complementary extinction azimuths of the nicol.

The sign of the vibration is, using the mean θ_c of the complementary settings of the compensator,

$$\begin{aligned} &\text{positive when} \quad 0 < (\theta_c - \theta_0) < \frac{\pi}{2} \\ &\text{negative when} \quad 0 > (\theta_c - \theta_0) > -\frac{\pi}{2} \end{aligned} \tag{31}$$

Finally the order ϕ of the compensator is obtained from

$$\cos\phi = \frac{\tan(c-n)}{\tan c\sqrt{1+\dfrac{\tan^2 2\omega_f}{\cos^2(c-n)}}} \tag{27}$$

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